Mathematics for Photonics Education

OP-TEC

Optics and Photonics Series

Mathematics for Photonics Education

Student Review and Study Guide

CORD

Leading Change in Education

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PREFACE

Over the past twenty-five years, CORD has had considerable experience in the laser/electro-optics area—developing curriculum, conducting workshops, and evaluating student achievement in self-help courses. Based on these experiences, CORD has learned that certain sets of mathematics skills in algebra, geometry, and trigonometry are essential for satisfactory progress in learning the technical content of material found in typical two-year photonics education programs. In general, students entering such programs come with varying backgrounds of mathematics achievement and varying retention levels of mathematics once learned.

Consequently, to help aspiring photonics technicians begin their studies with adequate math skills, CORD has developed a special text—*Mathematics for Photonics Education*. This text pulls together topics in algebra, geometry, and trigonometry as identified below:

- Scientific notation
- Unit conversion
- Introductory algebra
- Powers and roots
- Ratio and proportion
- Exponents and logarithms
- Graphing in rectangular coordinates
- Geometry
- Angle measures in two and three dimensions
- Trigonometry
- Special graphs

The material in each of these eleven sections begins with a list of *learning objectives* appropriate to the topical area of study. These are followed by a photonics "scenario," that is, a description of a *typical problem* that photonics technicians are likely to meet in their studies or in the workplace. The scenario is intended to demonstrate the need for the mathematics skills covered in the section and to motivate the student to acquire those skills. Next comes the core of the section—an *explanation of math concepts* along with worked-out *example problems* and *practice exercises*. Students are encouraged to work out the practice exercises and check their results against the answers provided. The explanatory sections close with solutions to the corresponding scenario questions—which at that point should be quite understandable to the students.

An associated assessment instrument containing multiple-choice questions and titled "Entering Student Assessment for *Mathematics for Photonics Education*" has been prepared to assist both students and teachers in determining the students' "mathematical readiness." Based on results of the assessment, the material in *Mathematics for Photonics Education* can be used to help entering students in several ways:

- The material can be presented as a whole in a "regular class" with teacher lecture and practice sessions, generally not for credit.
- The material can be assigned for appropriate students to "learn on their own" with occasional instructor monitoring and help.
- The material can be used "piecemeal" in a self-learning mode—with some instructor help—by students who require review and brushup in one or more specific mathematics areas.

SCIENTIFIC NOTATION

Objectives

When you have completed this lesson, you should be able to do the following:

1. Write decimal numbers in scientific notation
2. Write scientifically notated numbers in decimal notation

Photonics technicians need to use scientific notation to write and enter into a calculator or computer very large or very small numbers. For example, they will write the speed of light c as 3.0×10^{10} cm/s rather than 30,000,000,000 cm/s, and Planck's constant h as 6.625×10^{-34} J·s, rather than 0.0000000000000000000000000000000006625.

Photonics Scenario Involving Scientific Notation

Your supervisor asks you to determine the energy of a photon from an Nd:YAG laser. You know that the formula for the energy is:

$$E = \frac{hc}{\lambda}$$

where h Planck's constant (6.625×10^{-34} joule-seconds (J-s))

c Speed of light in a vacuum (2.998×10^{8} meter/second (m/s))

λ Wavelength (1.064×10^{-6} m for Nd:YAG laser photon)

E Energy in joules

Questions

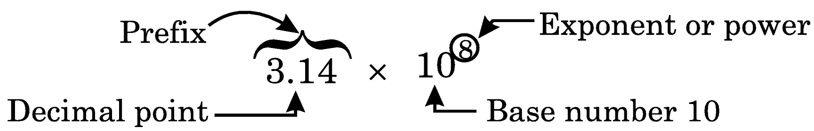
a. What energy in joules do you report to your supervisor?

b. Often, energy is given in terms of electron volts (eV). One eV is equal to 1.6×10^{-19} joules. What energy in eV do you tell your supervisor the Nd:YAG photon has?

Before looking at the solution, work through the lesson to further develop your skills in this area.

The Basics of Scientific Notation

Scientific notation makes use of "powers of ten." It is a convenient way to write large or small numbers. For example, the large number 314,000,000,000 becomes 3.14×10^{11} in scientific notation. The small number 0.000000314 becomes 3.14×10^{-7}. Before we learn how to write ordinary decimal numbers in scientific notation, let's learn a few definitions. Scientific notation involves a **prefix**, a **decimal point**, the **number 10**, and a number called an "**exponent**," or power. These are identified below.



Any one of the numbers in the prefix—3, 1, or 4—is referred to as a digit.

Scientific notation is based on the fact that you can write any number as a prefix times a string of tens. For example, a string of four tens—$10 \times 10 \times 10 \times 10$—can be written as 10^4. Study the following examples.

$$100 = 1 \times 100 = 1 \times (10 \times 10) = 1 \times 10^2$$

$$150 = 1.5 \times 100 = 1.5 \times (10 \times 10) = 1.5 \times 10^2$$

$$3750 = 3.75 \times 1000 = 3.75 \times (10 \times 10 \times 10) = 3.75 \times 10^3$$

$$37{,}500 = 3.75 \times 10{,}000 = 3.75 \times (10 \times 10 \times 10 \times 10) = 3.75 \times 10^4$$

Note that, in each case, a number (such as 3750) was written as a prefix (3.75) times a power of ten (10^3).

When a number is less than one, such as 0.1 or 0.015, it can also be written in scientific notation. For example:

$$0.1 = 1 \times \frac{1}{10} = 1 \times 10^{-1}; \quad \left(\frac{1}{10} = 10^{-1}\right)$$

$$0.01 = 1 \times \frac{1}{100} = 1 \times \frac{1}{10 \times 10} = 1 \times 10^{-2}; \quad \left(\frac{1}{10 \times 10} = \frac{1}{10^2} = 10^{-2}\right)$$

$$0.015 = 1.5 \times \frac{1}{100} = 1.5 \times \frac{1}{10 \times 10} = 1.5 \times 10^{-2}$$

Note that, in each case, a number less than one was written as a prefix times a power of ten. For a number less than one, the exponent or power used with the number 10 is always **negative**.

When a number such as 81,000 is written as 8.1×10^4, it's called "**scientific notation**." It's called that because scientists, engineers, and technicians often have to use very large or very small numbers. Writing one of these numbers as a prefix times a power of ten is easier and faster. To write numbers in scientific notation, follow these procedures.

Case 1—The general procedure for writing **numbers greater than one** (such as 81,000) in scientific notation is:

a. Locate the decimal point in the number.

81,000. ← (Original decimal point location)

b. Count the number of places to the left to shift the decimal point so that only **one digit** remains to the left of the new decimal point.

(Four places to the left)

8.1000. ← (Original decimal point location)

↑ (New decimal point location)

c. Use the number of places moved to the left (4) as the **positive exponent** in the power of ten (10) and write 81,000 in scientific notation as:

$$81{,}000 = 8.1 \times 10^4$$

Case 2—The general procedure for writing **numbers less than one** (such as 0.0715) in scientific notation is as follows:

a. Locate the decimal point in the number.

0.0715

↑ (Original decimal point location)

b. Count the number of places to the right to shift the decimal point so that only **one digit**, other than zero, remains to the left of the new decimal point.

(Two places to the right)

0 07.15

↑ (New decimal point location)

↑ (Original decimal point location)

c. Use the *number of places moved to the right* (2) as the **negative exponent** in the power of ten (10^{-2}) and write 0.0715 in scientific notation as:

$$0.0715 = 7.15 \times 10^{-2}$$

In Table 1, the procedures outlined above have been used to change ordinary decimal numbers to numbers in scientific notation. Look over each number and be sure that you agree with the conversion.

Table 1. Converting Numbers to Power-of-Ten Notation

375,000	$= 3.75 \times 10^5$
81,000	$= 8.1 \times 10^4$
623,000,000,000	$= 6.23 \times 10^{11}$
0.0715	$= 7.15 \times 10^{-2}$
0.0025	$= 2.5 \times 10^{-3}$
0.000000133	$= 1.33 \times 10^{-7}$

When a number such as 81,000 is changed to scientific notation, it's written as 8.1×10^4. Other forms of power-of-ten notation, equally correct and equally useful but not written in scientific notation, are the following:

$$\begin{aligned} 81{,}000 &= 81 \times 10^3 \\ &= 810 \times 10^2 \\ &= 8100 \times 10^1 \\ &= 0.81 \times 10^5 \end{aligned}$$

Similarly, for numbers less than one, more than one form for the power-of-ten notation is possible. For example:

$$\begin{aligned} 0.0025 &= 2.5 \times 10^{-3} \\ &= 25 \times 10^{-4} \\ &= 250 \times 10^{-5} \\ &= 0.25 \times 10^{-2} \end{aligned}$$

A specific form of power-of-ten notation that is sometimes used is called **engineering notation**, though it is not common as scientific notation. It is important to understand the differences between the two. For engineering notation, the exponent must be a multiple of 3 (…, –12, –9, –6, –3, 3, 6, 9, 12, …) or zero, whereas for scientific notation it is necessary only that the exponent be an integer (…, –3, –2, –1, 0, 1, 2, 3, …). In Table 2 are examples of numbers expressed both in scientific notation and engineering notation.

Table 2. Equivalent Numbers in Scientific Notation and Engineering Notation

	Scientific Notation	Engineering Notation
a	7.2×10^6	7.2×10^6
b	3.5×10^4	35×10^3
c	8.4×10^8	840×10^6
d	9.1×10^{-5}	91×10^{-6}
e	6.0×10^{-7}	600×10^{-9}
f	4.2×10^2	420×10^0
g	5.7×10^{-21}	5.7×10^{-21}

Do you see the pattern? When an exponent is not a multiple of 3 or a zero (as in letters b–f above), it is rounded down to the next multiple of 3 or zero. If moved, it will always be either one (b, d) or two places down (c, e, f). Then it is necessary to move the decimal place of the prefix (number before the "×") to the right the same number of times as places moved down in the previous step. So, for letters b and d, the decimal in the prefix is moved one place to the right (essentially multiplying the prefix by 10) and, for letters c, e, and f, the decimal is moved two places to the right (like multiplying by 100). Once the number is in engineering notation, the prefix will always fall in the range: $1 \leq$ prefix < 1000.

It is also quite simple to use power-of-ten notation on a calculator. For example, an entry of 9.87×10^{-6} would look like this: [9] [.] [8] [7] [E] [–] [6]. The "E" stands for the "× 10" and also is commonly seen as "Exp" or "EE." The "–" is a negative sign, not a minus sign. Some calculators have buttons with "(–)" or "+/–."
A calculator also reports extremely large or small numbers in scientific notation in the same form used to enter them. For example, when calculating 123^{45}, using the y^x key, a calculator will show 1.11E94. The benefits of scientific notation are clear in this problem: Scientific notation saves the user from having to write out the entire number (95 digits!).

Solutions to the Scenario Questions

Following are the solutions to the questions posed under "Photonics Scenario Involving Scientific Notation."

a. $E = \dfrac{hc}{\lambda}$

$$E = \frac{(6.625 \times 10^{-34}\,\text{J} \cdot \cancel{\text{s}})(2.998 \times 10^{8}\,\cancel{\text{m}}/\cancel{\text{s}})}{1.064 \times 10^{-6}\,\cancel{\text{m}}}$$

$E = 1.87 \times 10^{-19}$ J

b. $$E = \frac{1.87 \times 10^{-19}\,\cancel{\text{J}}}{} \Bigg| \frac{1\text{ eV}}{1.6 \times 10^{-19}\,\cancel{\text{J}}} = 1.17\text{ eV}$$

Practice Exercises

Exercise 1

The following numbers are all larger than one. Change each to **scientific notation**—with only *one* digit remaining to the left of the decimal point in the final answer.

Example: $3860 = 3.86 \times 10^3$

a. 38,600 = ____________________

b. 157,300 = ____________________

c. 300,000,000 = ____________________

d. 147 = ____________________

e. 93,000,000 = ____________________

Exercise 2

The following numbers are all less than one. Change each to **scientific notation**—with only *one* digit (other than zero) remaining to the left of the decimal point in the final answer.

Example: $0.015 = 1.5 \times 10^{-2}$

a. 0.0036 = ____________________

b. 0.715 = ____________________

c. 0.000025 = ____________________

d. 0.002 = ____________________

e. 0.00083 = ____________________

Exercise 3

The following numbers are written in power-of-ten notation. Change each to decimal notation.

Examples: $8.36 \times 10^{-1} = 0.836$
$3.01 \times 10^3 = 3010$

a. 81.5×10^{-1} = ____________________

b. 47.71×10^{-4} = ____________________

c. 326.1×10^{-4} = ____________________

d. 4.771×10^4 = ____________________

e. 389×10^{-5} = ____________________

f. 3×10^8 = ____________________

Exercise 4

Change the following numbers to power-of-ten notation by filling in the correct prefixes.

Example: $3860 = 3.86 \times 10^3$

a. $38{,}600 =$ ____________________ $\times 10^2$

b. $157{,}300 =$ ____________________ $\times 10^4$

c. $23{,}600 =$ ____________________ $\times 10^5$

d. $0.00147 =$ ____________________ $\times 10^{-3}$

e. $0.056 =$ ____________________ $\times 10^{-2}$

f. $0.0791 =$ ____________________ $\times 10^{-3}$

g. Indicate which of the numbers above have been rewritten in scientific notation.

Exercise 5

Multiply or divide the expressions below with a calculator then write the answers in **scientific notation**.

a. $(1.25 \times 10^{12}) + (8.7 \times 10^{11}) =$ ____________________

b. $(3.2 \times 10^{6}) \times (5.9 \times 10^{4}) =$ ____________________

c. $26 \div (4.6 \times 10^{14}) =$ ____________________

d. $(4.9 \times 10^{-32}) \times (1.6 \times 10^{33}) =$ ____________________

e. $12^{45} =$ ____________________

Solutions to Practice Exercises

1. **a.** 3.86×10^4
b. 1.573×10^5
c. 3×10^8
d. 1.47×10^2
e. 9.3×10^7

2. **a.** 3.6×10^{-3}
b. 7.15×10^{-1}
c. 2.5×10^{-5}
d. 2×10^{-3}
e. 8.3×10^{-4}

3. **a.** 8.15
b. 0.004771
c. 0.03261
d. 47710
e. 0.00389
f. 300,000,000

4. **a.** 386
b. 15.73
c. 0.236
d. 1.47
e. 5.6
f. 79.1
g. d and e

5. **a.** 2.12×10^{12}
b. 1.89×10^{11}
c. 5.65×10^{-14}
d. 7.84×10^{1}
e. 3.66×10^{48}

UNIT CONVERSION

Objectives

When you have completed this section, you should be able to do the following:

1. Multiply and divide numbers that contain units and obtain answers that correctly express numbers and units
2. Express numerical values and units (such as 3.4×10^{-3} meters) in prefix notation (3.4 millimeters)
3. Given an equation with several terms, check the dimensions for each item and then check the equation as a whole
4. Convert a physical quantity expressed in one set of units to an equivalent quantity in a different set of units

Photonics technicians need to work with mathematical expressions that involve many different units of measurement, both in the English and metric system. Frequently they need to convert between units in different systems, such as working with the speed of light as either 186,000 mi/sec or 3.00×10^{10} cm/sec, or expressing a wavelength as 0.555 μm, 555 nm, or 5550 Å.

Photonics Scenario Involving Units

The heat transfer rate in a laser can be found from the equation:

$$\frac{\Delta Q}{\Delta t} = \frac{\Delta m}{\Delta t} \times c \times \Delta T$$

where $\frac{\Delta Q}{\Delta t}$ Heat transfer rate

$\frac{\Delta m}{\Delta t}$ Flow rate of the coolant in grams per second (g/s)

c Specific heat of the coolant in calories per gram per Celsius degree of temperature change $\left(\frac{\text{cal}}{\text{g} \bullet {}^\circ\text{C}}\right)$

ΔT Temperature change of the coolant in degrees Celsius (°C)

You make measurements on the heat transfer rate for a coolant flowing around the gain medium of an operating laser and find that $(\Delta m/\Delta t) = 33.3$ g/s and $\Delta T = 17.7$ °C. You know that the specific heat for water is 1 cal/(g • °C).

Question

What value and units do you note in your lab book for the heat transfer rate in this experiment?

Before looking at the solution, work through the lesson to further develop your skills in this area.

The Basics of Units

Part 1

Most quantities used and measured by technicians are described by numbers that include units. For example, consider the following:

a. 10 feet: 10 is the number; feet is the unit.

b. 20 miles/hour: 20 is the number; miles/hour is the unit.

c. 3 cm^3: 3 is the number; cm^3 is the unit.

d. 5 lb/ft^2: 5 is the number; lb/ft^2 is the unit.

When numbers that contain units are multiplied or divided, the numbers and units are handled separately. Study the following examples:

a. $10 \text{ ft} \times 20 \text{ ft} = (10 \times 20) \times (\text{ft} \times \text{ft}) = 200 \text{ ft}^2$.
(Multiplying ft × ft is feet squared, written as ft^2.)

b. $$\left(20 \ \frac{\text{mi}}{\text{h}}\right) \times 10 \text{ h} = (20 \times 10) \times \left(\frac{\text{mi}}{\cancel{\text{h}}} \times \cancel{\text{h}}\right) = 200 \text{ mi}$$

(Multiplying $\frac{\text{mi}}{\text{h}} \times \text{h}$ is simplified by canceling the "h" that occurs in both the numerator and the denominator.)

c. $$\left(5 \ \frac{\text{lb}}{\text{ft}^2}\right) \times 10 \text{ ft}^2 = (5 \times 10) \times \left(\frac{\text{lb}}{\cancel{\text{ft}^2}} \times \cancel{\text{ft}^2}\right) = 50 \text{ lb}$$

(The "ft^2" in the numerator cancels "ft^2" in the denominator.)

d. $$5 \text{ g} \div 10 \text{ cm}^2 = \frac{5 \text{ g}}{10 \text{ cm}^2} = \left(\frac{5}{10}\right) \times \left(\frac{\text{g}}{\text{cm}^2}\right) = 0.5 \text{ g/cm}^2$$

e. $$20 \text{ N} \div 5 \text{ m}^2 = \frac{20 \text{ N}}{5 \text{ m}^2} = \left(\frac{20}{5}\right) \times \left(\frac{\text{N}}{\text{m}^2}\right) = 4 \text{ N/m}^2$$

f. $$5 \ \frac{\text{lb}}{\text{ft}^3} \times 2 \text{ ft} = (5 \times 2) \times \left(\frac{\text{lb} \times \text{ft}}{\text{ft}^3}\right) = 10\left(\frac{\text{lb} \times \cancel{\text{ft}}}{\cancel{\text{ft}} \times \text{ft} \times \text{ft}}\right) = 10 \text{ lb/ft}^2$$

Part 2

The prefixes given in Table 3 are defined for numbers given in powers of ten. In the table, the most important prefixes are indicated by asterisks(*). Begin by learning the most important ones. Then review the others often until you recognize them.

Example: Use Table 3 to see that the SI (International System of Units) prefixes and units for the numbers given below are correct.

100×10^{-2} meters = 100 *centimeters* (cm)

1×10^{3} meters = 1 *kilometer* (km)

5×10^{-3} meters = 5 *millimeters* (mm)

1.8×10^{-6} seconds = 1.8 *microseconds* (μs)

2×10^{12} watts = 2 *terawatts* (TW)

1×10^{3} cal = 1 *kilocalories* (kcal)

0.2×10^{6} newtons = 0.2 *meganewtons* (MN)

Table 3. SI Prefixes

Factor by Which the Unit Is Multiplied	Prefix	Symbol
10^{12}	tera	T
*10^{9}	giga	G
*10^{6}	mega	M
*10^{3}	kilo	k
10^{2}	hecto	h
$10^{1} = 10$	deca	da
10^{-1}	deci	d
*10^{-2}	centi	c
*10^{-3}	milli	m
*10^{-6}	micro	μ
*10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a

*Most commonly used

Part 3

Operations performed with units are similar to operations performed with numbers. Therefore, if the units that accompany a numerical answer aren't correct, an error has been made in the problem-solving process. To make this point more clear, look at the following equation:

$$S = vt + \tfrac{1}{2}\,at^2$$

where S = Distance
v = Velocity (distance/time)
t = Time
a = Acceleration (distance/time^2)

This equation involves solving for distance (S). Therefore, each term, such as "vt" and "$\tfrac{1}{2}\,at^2$" on the right side of the equation, must also have units of "distance." Let's show this by first substituting the correct units for v, t, and a on the right side, then canceling units where appropriate.

$$S = vt + \tfrac{1}{2}\,at^2 \qquad \text{(Ignore constants, such as } \tfrac{1}{2}\text{, that carry no units.)}$$

$$S = \left(\frac{\text{distance}}{\cancel{\text{time}}} \times \cancel{\text{time}}\right) + \left(\frac{\text{distance}}{\cancel{\text{time}^2}} \times \cancel{\text{time}^2}\right) \qquad \text{(Cancel time units.)}$$

$$S = \text{distance} + \text{distance}$$

$$S = \text{distance}$$

The equation is correct. That's because a **distance** plus a **distance** does equal a distance, and we know that S should be a distance unit.

Now let's substitute SI units in the equation and show the same thing. Use t in seconds, v in meters/s, and a in meters/s^2. Then:

$$S = vt + \tfrac{1}{2}\,at^2$$

$$S = \left(\frac{\text{meters}}{\cancel{\text{s}}} \times \cancel{\text{s}}\right) + \left(\frac{\text{meters}}{\cancel{\text{s}^2}} \times \cancel{\text{s}^2}\right) \qquad \text{(Cancel s units.)}$$

$$S = \text{meters} + \text{meters}$$

$$S = \text{meters}$$

The equation is dimensionally correct. Since the left side S is a distance, it should have "distance" units. It does; the right side is in meters (a distance unit).

Part 4

Just as apples and oranges don't add up, neither do numbers that carry different units. For example, "2 inches + 1 foot" doesn't equal either 3 inches or 3 feet. To add 2 inches to 1 foot, you must **first** express both terms in the **same** units.

Thus, if you change 1 foot to 12 inches and then add "2 inches + 12 inches," you get a correct answer: 14 inches. So, "2 inches + 1 foot" = 14 inches. If you change 2 inches to $\frac{1}{6}$ foot and then add "$\frac{1}{6}$ ft + 1 ft," you will get another correct answer: $1\frac{1}{6}$ ft.

Suppose you are asked to add the following:

1 meter + 40 centimeters = ___________

Since the units aren't alike, you can't add the two numbers as they stand. But you can change 1 meter to 100 centimeters and then add.

100 cm + 40 cm = 140 cm (a correct answer)

Or, you can change 40 cm to 0.40 meter (since 40 cm equals $\frac{40}{100}$ of a meter, and $\frac{40}{100}$ = 0.40) and then add.

1 m + 0.40 m = 1.40 m (also a correct answer)

To convert units from one form to another, such as feet to inches or kilograms to grams, follow the general conversion process shown in Figure 1. The letters A, B, and C represent numbers.

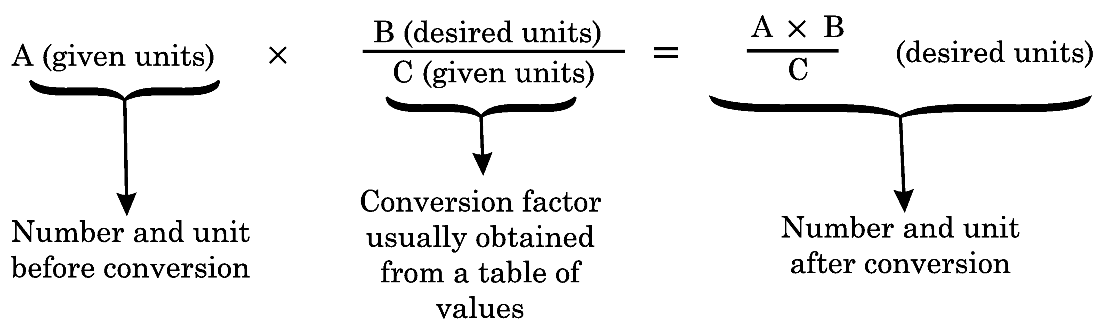


Fig. 1 Unit-conversion process

Let's try some examples that show how the conversion process outlined in Figure 1 is applied.

Example 1

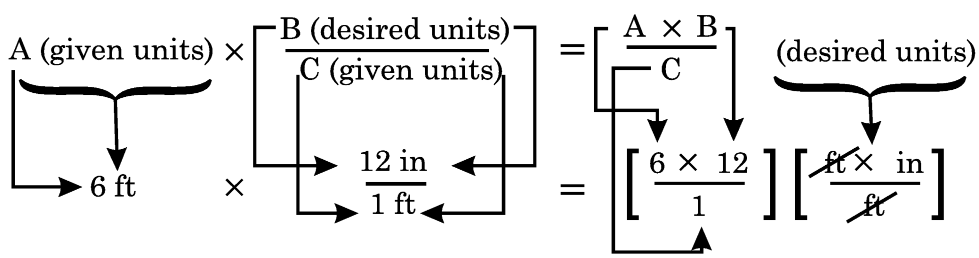
Given: A length of 6 feet

Find: The same length in inches

Solution: From a table of conversion values, we find that 1 ft = 12 in. Therefore, we can form the conversion factor $\frac{\text{B (desired units)}}{\text{C (given units)}}$ as follows:

$$\frac{12\text{ in}}{1\text{ ft}}$$

The ratio is equal to 1 because numerator and denominator are equal lengths. (**Note:** Multiplying something by 1 doesn't change it.) Now follow the process shown in Figure 1.



Thus,

$$6\text{ ft} \times \frac{12\text{ in}}{1\text{ ft}} = \left(\frac{6 \times 12}{1}\right)\left(\frac{\cancel{\text{ft}} \times \text{in}}{\cancel{\text{ft}}}\right) = 72\text{ in}$$

Thus, the result is 6 ft = 72 in.

Note: In the conversion process of Example 1, we multiplied 6 ft by the ratio $\frac{12\text{ in}}{1\text{ ft}}$. The ratio is equal to 1. Thus, the length of 6 ft wasn't changed, since it was multiplied by 1. It was only converted to an equivalent length expressed in inches—72 inches.

Example 2

Given: A distance of 10 miles

Find: The same distance in feet

Solution: From a conversion table for lengths, we find that:

$$1\text{ mile} = 5280\text{ feet}$$

With this equality we can form two conversion ratios, each equal to 1.

$$\frac{1\text{ mile}}{5280\text{ ft}} \text{ and } \frac{5280\text{ ft}}{1\text{ mile}} \Rightarrow \frac{\text{B (desired units)}}{\text{C (given units)}}$$

Since we're converting from miles (given units) to feet (desired units), we'll use the ratio $\frac{5280 \text{ ft}}{1 \text{ mile}}$ for the conversion factor. Then, using the process shown in Figure 1, multiply and cancel units.

$$10 \text{ mi} \times \left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right) = \left(\frac{10 \times 5280}{1}\right)\left(\frac{\cancel{\text{mi}} \bullet \text{ft}}{\cancel{\text{mi}}}\right) = 52{,}800 \text{ ft}$$

Therefore, 10 miles = 52,800 feet.

Example 3

Given: A distance of 52,800 ft

Find: The same distance in inches

Solution: From a conversion table, we find that 1 foot = 12 inches. Since we're changing feet (given units) to inches (desired units), we use the conversion ratio $\frac{12 \text{ in}}{1 \text{ ft}}$. Multiply and cancel units.

$$52{,}800 \text{ ft} \times \left(\frac{12 \text{ in}}{1 \text{ ft}}\right) = \left(\frac{52{,}800 \times 12}{1}\right)\left(\frac{\cancel{\text{ft}} \bullet \text{in}}{\cancel{\text{ft}}}\right) = 633{,}600 \text{ in}$$

Therefore, 52,800 ft = 633,600 in.

If we combine the results in Examples 2 and 3, we find that 10 miles = 633,600 inches. We found this in two steps. We could have done it in one step. See Example 4.

Example 4

Given: A distance of 10 miles

Find: The same distance in inches

Solution: From a table of conversions between lengths, we are given:

1 mi = 5280 ft and 1 ft = 12 in.

We form the conversion ratios $\left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right)$ and $\left(\frac{12 \text{ in}}{1 \text{ ft}}\right)$.

Then multiply and cancel units.

$$10 \text{ mi} \times \left(\frac{5280 \text{ ft}}{1 \text{ mi}}\right) \times \left(\frac{12 \text{ in}}{1 \text{ ft}}\right) = \left(\frac{10 \times 5280 \times 12}{1 \times 1}\right)\left(\frac{\cancel{\text{mi}} \bullet \cancel{\text{ft}} \bullet \text{in}}{\cancel{\text{mi}} \bullet \cancel{\text{ft}}}\right).$$

Therefore, 10 mi = 633,600 in.

The answer is the one we found in two steps in Examples 2 and 3.

Solution to the Scenario Question

Following is the solution to the question posed under "Photonics Scenario Involving Units."

Use the given equation and substitute in numbers with given values and units:

$$\frac{\Delta Q}{\Delta t} = \frac{\Delta m}{\Delta t} \times c \times \Delta T$$

$$\frac{\Delta Q}{\Delta t} = 33.3\ \frac{\cancel{\text{g}}}{\text{s}} \times 1\ \frac{\text{cal}}{\cancel{\text{g}} \bullet \cancel{°\text{C}}} \times 17.7\ \cancel{°\text{C}}$$

$$\frac{\Delta Q}{\Delta t} = 589.4\ \frac{\text{cal}}{\text{s}}$$

The heat transfer rate for this laser and associated cooling system is about 590 cal/s.

Practice Exercises

Multiply or divide the quantities given below as indicated, and obtain each answer as a correct number and unit.

Exercise 1

a. 5 lb × 10 ft = ____________________

b. 10 lb ÷ 2 ft^2 = ____________________

c. 100 N ÷ 5 m^2 = ____________________

d. $1000\ \text{cm}^3 \times 13.6\ \frac{\text{g}}{\text{cm}^3} =$ ____________________

e. $16\ \frac{\text{miles}}{\text{hour}} \times 0.5\ \text{hour} =$ ____________________

f. $5\ \text{hours} \times 60\ \frac{\text{minutes}}{\text{hour}} =$ ____________________

g. $62.4\ \frac{\text{lb}}{\text{ft}^3} \times 10\ \text{ft} =$ ____________________

h. $10\ \frac{\text{N}}{\text{m}^3} \times 2\ \text{m} =$ ____________________

Exercise 2

Place the names and symbols for the following factors in the spaces provided. The answer to the first one is provided.

Factor	Name	Symbol
10^{3}	kilo	k
10^{-3}		
10^{6}		
10^{-6}		
10^{-2}		
10^{9}		
10^{-9}		
10^{-12}		

Check answers by comparing to Table 3.

Exercise 3

For each value given below, change the power-of-ten notation to a number. Then add the correct prefix symbol to the unit.

5.6×10^{3} meters	=	5.6 kilometers (km)
6.8×10^{3} calories	=	
10×10^{-2} meters	=	
5×10^{-3} meters	=	
3.14×10^{-9} seconds	=	

Exercise 4

Use Table 3 to identify the correct prefixes for the power-of-ten numbers given below.

10^{-3} meters	=	
10^{-3} grams	=	
10^{-6} calories	=	
10^{-2} meters	=	

Exercise 5

Rewrite “15,600 grams” using power-of-ten notation and a unit prefix name that involves kilograms.

Exercise 6

Given the equation:

$$v_f = v_i + at$$

where v_f = Speed (ft/s or m/s)
v_i = Initial speed (ft/s or m/s)
a = Acceleration (ft/s^2 or m/s^2)
t = Time (s)

Substitute the proper units for v_i, a, and t, first in the English system and then in SI. Show that each term has dimensions of ft/s or m/s, hence speed—the correct units for v_f.

Exercise 7

Given the equation:

$$R_E = \frac{\rho \ell}{A}$$

where R_E = Electrical resistance in ohms (Ω)
ρ = Electrical resistivity in ohm • cm
ℓ = Length in cm
A = Cross-sectional area in cm^2

Substitute the proper units for ρ, ℓ, and A into the equation. Show that the terms on the right reduce to ohms (Ω). This is the correct unit for the resistance R_E on the left side.

Exercise 8

Convert 150 centimeters to an equal length in millimeters. A table of conversions gives 1 cm = 10 mm. This solution is partially set up as follows:

$$150 \text{ cm} \times \frac{10 \text{ mm}}{1 \text{ cm}} =$$

Exercise 9

Convert 21 liters to an equal volume in gallons. A table of conversions gives 1 gal = 3.785 liters.

Exercise 10

Convert 40 centimeters to an equal length in meters. A table of conversions gives 1 m = 100 cm.

Exercise 11

Convert 5 hours to an equal time in seconds. A table of conversions gives 1 h = 3600 s.

Exercise 12

Convert 10.5 mi/h to an equal speed in ft/s. A table of conversions gives 1 mi = 5280 ft and 1 h = 3600 s. Solve this problem in one step, as was done in Example 4.

Solutions to Practice Exercises

1. **a.** 50 lb • ft

   b. 5 lb/ft^2

   c. 20 N/m^2

   d. 13,600 g

   e. 8 miles

   f. 300 minutes

   g. 624 lb/ft^2

   h. 20 N/m^2

2. See Table 3.

3. 6.8 kilocalories (kcal)
   10 centimeters (cm)
   5 millimeters (mm)
   3.14 nanoseconds (ns)

4. millimeters (mm)
   milligrams (mg)
   microcalories (μcal)
   centimeters (cm)

5. 1.56×10^1 kg

6. $v_f = v_i + at$

$$\frac{\text{ft}}{\text{s}} = \frac{\text{ft}}{\text{s}} + \left(\frac{\text{ft}}{\text{s}^{\cancel{2}}}\right)(\cancel{\text{s}})$$

$$\frac{\text{ft}}{\text{s}} = \frac{\text{ft}}{\text{s}}$$

$v_\text{f} = v_\text{i} + at$

$$\frac{\text{m}}{\text{s}} = \frac{\text{m}}{\text{s}} + \left(\frac{\text{m}}{\text{s}^{\cancel{2}}}\right)(\cancel{\text{s}})$$

$$\frac{\text{m}}{\text{s}} = \frac{\text{m}}{\text{s}}$$

7. $R_\text{E} = \frac{\rho \ell}{A}$

$$\Omega = \frac{(\Omega \bullet \cancel{\text{cm}})\cancel{\text{cm}}}{\cancel{\text{cm}^2}}$$

$$\Omega = \Omega$$

8.

$$\frac{150\ \cancel{\text{cm}}}{} \left| \frac{10\ \text{mm}}{1\ \cancel{\text{cm}}} \right. = 1500\ \text{mm}$$

9.

$$\frac{21\ \cancel{L}}{} \left| \frac{1\ \text{gal}}{3.785\ \cancel{L}} \right. = 5.55\ \text{gal}$$

10.

$$\frac{40\ \cancel{\text{cm}}}{} \left| \frac{1\ \text{m}}{100\ \cancel{\text{cm}}} \right. = 0.4\ \text{m}$$

11.

$$\frac{5\ \cancel{\text{h}}}{} \left| \frac{3600\ \text{s}}{1\ \cancel{\text{h}}} \right. = 18{,}000\ \text{s}$$

12.

$$\frac{10.5\ \cancel{\text{mi}}}{\cancel{\text{h}}} \left| \frac{5280\ \text{ft}}{1\ \cancel{\text{mi}}} \right| \frac{1\ \cancel{\text{h}}}{3600\ \text{s}} = 15.4\ \text{ft/s}$$

INTRODUCTORY ALGEBRA

Objectives

When you complete this lesson, you should be able to do the following:

1. Simplify an expression (order of operations)
2. Rearrange formulas so that one variable is isolated
3. Substitute into formulas and solve for unknown quantities

Photonics technicians need to work with many different algebraic expressions that involve variables and constants, simplifying them, solving for a desired variable, and substituting known values for given quantities. For example, they need to know how to solve for the wavelength λ in an expression such as $\Delta E = \frac{hc}{\lambda}$, or for the mirror curvature r_1 in an equation such as $g_1 = 1 - \frac{L}{r_1}$.

Photonics Scenario Involving Introductory Algebra

You are measuring the characteristics of an argon-ion laser with a cavity length (distance between the mirrors) of 50 cm. The gain medium fills the space between the same two mirrors. The reflectivity of the HR (high-reflectivity mirror) is 0.998. The reflectivity of the output mirror is 0.9575. You determine that the round trip gain (loop gain) is 0.969 for a round trip cavity loss of 8.0%. You want to calculate the amplifier gain G_A of the laser with the following equation:

$$G_L = R_1 R_2 G_A^2 (1 - \alpha)$$

where
- G_L Loop gain
- R_1 Reflectivity of HR mirror
- R_2 Reflectivity of output mirror
- G_A Amplifier gain
- α Round trip cavity loss

Question

What do you find for the amplifier gain, G_A?

Before looking at the solution, work through the lesson to further develop your skills in this area.

Formulas

What is a formula? In mathematics a formula is a way of using symbols to write a sentence. The following examples show formulas followed by the complete sentences they represent.

Formula: $p = 2l + 2w$

Sentence: The perimeter of a rectangle is the sum of twice the length and twice the width.

Formula: $i = prt$

Sentence: The simple interest is the product of the principal, the annual rate, and the time in years.

Which do you think is easier to read and remember—the formula (written in mathematics symbols) or the sentence (written in words)?

What does a formula do? A formula shows how some quantities relate to each other. For example, $C = \pi d$ shows that the circumference of a circle is always equal to the diameter of the circle multiplied by the number π (approximately 3.1416). Another way to express this relationship is to say, "The distance *around* a circle is always roughly three times the distance *across* the circle."

A formula is often the shortest, easiest way to write the relationship between two or more quantities. Once a formula is written correctly, it is ready to be used. Always take care to substitute correct numbers and units for known symbols in an equation. Only then can you calculate the number and unit of the unknown (isolated) symbol correctly.

Order of Operations

It is important to remember a particular order of operations when simplifying the expressions on either side of the equal sign. The mathematical operations are *addition*, *subtraction*, *multiplication*, and *division*. A combination of numbers and operations, such as $130 \div 10 + 3 \times 2$, is called a numerical expression. An algebraic expression contains one or more variables. For example, $2x^2 - 5x + 3$ is an algebraic expression.

To evaluate an expression means to find its value. When you evaluate an algebraic expression you replace the variable with a value. Rules must be followed so that everyone knows which operation to perform first and to ensure that everyone finds the same answer when evaluating an expression. The rules are called the **order of operations**.

1. **P**arentheses—Evaluate all operations inside parentheses and brackets (grouping symbols).
2. **E**xponents—Evaluate all exponents and powers.

3. **M**ultiplication and **D**ivision—Multiply and divide from left to right.
4. **A**ddition and **S**ubtraction—Add and subtract from left to right.

Helpful reminder: **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally (PEMDAS)

Example 1

Simplify:	$8 - 3(5 + 2^4)$	
Solution:	8 – 3(5 + 16)	Inside parenthesis, do exponent first
	8 – 3(21)	Next finish parentheses with addition
	8 – 63	Then multiplication
	–55	Subtraction last

Example 2

Given: The formula $V = I \times R$

V = 10 volts (V) and R = 2 ohms (Ω).

Find: Current, I

Solution: The equation isn't in the form with I isolated. So isolate I.

$$V = I \times R$$

$$\frac{V}{R} = \frac{I \times \not{R}}{\not{R}}$$ (Divide both sides by R, then cancel R's.)

$$\frac{V}{R} = I$$

$$I = \frac{V}{R}$$ (Reverse the order of the equation.)

Now, with I isolated, substitute in the given values of V and R.

$$I = \frac{10\text{ V}}{2\ \Omega} = \left(\frac{10}{2}\right) \times \left(\frac{V}{\Omega}\right) = 5\left(\frac{V}{\Omega}\right) \quad \left(1\frac{V}{\Omega} \equiv 1\text{ A}\right)$$

$$I = 5\text{ A}$$

The current is 5 amperes.

Example 3

Rearrange the formula for the volume of a sphere so that you can isolate and solve for the radius r.

$$V = \tfrac{4}{3}\pi r^3$$

First, ask yourself this question: "What must I do to each side of the equation to free the term r^3?"

In this formula, r^3 is multiplied by 4, multiplied by π, and divided by 3. So, if you divide by 4, divide by π, and multiply by 3, you will have only r^3 left. Let's go through the steps carefully.

Notice that r^3 becomes isolated in the product $\left(\frac{4}{3}\pi r^3\right)$ if you do the following:

- Divide by 4
- Divide by π
- Multiply by 3

But dividing by 4, dividing by π, and multiplying by 3 is the same as multiplying by $\frac{3}{4\pi}$, all at once. Apply this same operation to *each* side of the equation, as follows.

$$\left(\frac{3}{4\pi}\right) \times V = \left(\frac{\cancel{3}}{\cancel{4\pi}}\right)\left(\frac{\cancel{4\pi}}{\cancel{3}} r^3\right)$$

The terms on the right simplify to r^3, since $\frac{3}{4\pi} \times \frac{4\pi}{3} = 1$. The formula that has been rearranged becomes:

$$\frac{3V}{4\pi} = r^3$$

You can rewrite this equation with r^3 on the left.

$$r^3 = \frac{3V}{4\pi}$$

But r^3 is not what you want. You want r. To get r from r^3, you have to take the cube root. So, taking the cube root of *each* side, you isolate r, obtaining for the final equation:

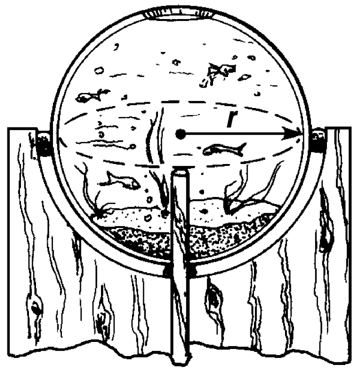
$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

This is the formula that helps you find the radius of a sphere if you know the volume of the sphere.

For a given problem, if you know the volume V, you would use your calculator to enter the expression $\frac{3V}{4\pi}$ and then use the *cube root* key or the $\sqrt[x]{y}$ key to get the answer for the radius r. Let's try that in the following example.

Example 4

A spherical tank must hold 6 cubic feet of liquid. What is the tank's radius to the nearest tenth of a foot?



Rewrite the formula, substituting the number values that you know for the variables.

$$r = \sqrt[3]{\frac{3V}{4\pi}}$$

$$r = \sqrt[3]{\frac{3(6 \text{ ft}^3)}{4\pi}}$$

Now, solve for r with the help of your calculator. You should find that $r = 1.1$ ft (rounded).

If you didn't get the answer $r = 1.1$ ft, try again. If you still don't, you can check your use of the calculator against the following steps. (Remember, there's always more than one way to get the answer. Your method may be better and shorter than the one outlined below.)

One way to solve for r in the formula $r = \sqrt[3]{\frac{3(6 \text{ ft}^3)}{4\pi}}$:

Clear the calculator
Enter 3
Press ×
Enter 6
Press ÷
Enter 4
Press ÷
Press the π key
Press = (display should show 1.4323945)
Press INV or 2nd Function key
Press y^x key
Press 3 (for cube root)

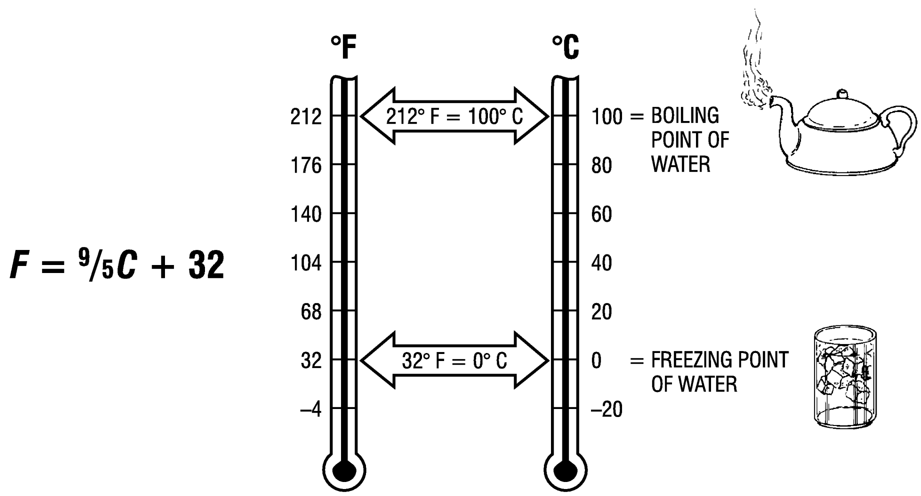
Press = (not needed on some calculators)
Answer 1.1272517 appears in window.
Round to the nearest tenth to get $r = 1.1$ ft.

Thus, to have a volume of 6 ft^3, you've found that a sphere must have a radius of about 1.1 ft.

If you want to check that answer, put $r = 1.1$ ft back in the original formula for the volume of a sphere, $V = \frac{4}{3}\pi r^3$. Should you expect to get $V = 6$ ft^3 exactly? Why? What value should you use for r to get $V = 6$ ft^3 as closely as possible?

* * *

Now let's look at the formula that shows the relationship between the Fahrenheit and Celsius temperature scales, as illustrated in the figure below.



The formula to convert a temperature on the Celsius scale (°C) to the equivalent temperature on the Fahrenheit scale (°F) looks like this:

$$F = \frac{9}{5}C + 32$$

Notice that the constant 32 is added to the expression $\frac{9}{5}C$.

Example 5

Rewrite this formula so you can use it to change a Fahrenheit temperature to one on the Celsius scale. Begin with the given formula:

$$F = \frac{9}{5}C + 32$$

(**Note:** F, C, and 32 are generally written as °F, °C, and 32°—with the degree symbol attached. We have, however, omitted the degree symbol to keep the formula as simple as possible.)

$$F = \frac{9}{5}C + 32$$

Since "32" is added in the equation, *subtract* 32 from each side to free the term $\frac{9}{5}C$—the term that contains C:

$$F - 32 = \frac{9}{5}C + 32 - 32$$

After you simplify the right side, you get

$$F - 32 = \frac{9}{5}C$$

Now multiply the right side by 5 and divide by 9 to "free" the C of the coefficient $\frac{9}{5}$. That's the same as multiplying by the fraction $\frac{5}{9}$. Do the same thing to the left side.

$$\frac{5}{9}\,(F - 32) = \frac{\cancel{5}}{\cancel{9}}\left(\frac{\cancel{9}}{\cancel{5}}C\right)$$

Notice that the multiplying fraction $\frac{5}{9}$ *must multiply the entire left side and the entire right side*. So put parentheses around each side before you multiply.

After you divide the fives and nines on the right side, you have this equation:

$$\frac{5}{9}(F - 32) = C$$

or

$$C = \frac{5}{9}(F - 32)$$

Remember, a number or letter written next to parentheses means that you multiply what is in the parentheses by that number or letter.

Thus, the expression $\frac{5}{9}(F - 32)$ means that you calculate $(F - 32)$ and multiply that result by $\frac{5}{9}$.

You have learned how to begin with a formula such as $F = \frac{9}{5}C + 32$ and rearrange it to isolate the variable C—to get the formula $C = \frac{5}{9}(F - 32)$. You can use the "rearranged" formula if you know the Fahrenheit temperature F and you want to calculate the corresponding Celsius temperature C.

Example 6

Find what 36°F is on the Celsius scale.

First, choose the version of the formula that has the variable you are looking for isolated on one side of the equal sign.

Since you are looking for a temperature in °C, you would choose the second version of the formula, with C isolated on the left side:

$$C = \frac{5}{9}(F - 32)$$

Substitute the number values from the problem for the variables you know in the formula. You are given that the Fahrenheit temperature is 36°. Therefore,

$$C = \frac{5}{9}(36^\circ - 32^\circ)$$

Use your calculator to solve this formula for C. If you round your answer to two decimal places and attach the proper units, you should get $C = 2.22°C$. How did you do? Did you use the parenthesis keys? Can you solve for C without using the parenthesis keys?

Some problems require more algebraic steps to come to solutions. For example, solve for q in the following equation:

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

where f = Focal length of the lens
p = Distance of the object being viewed from the center of the lens
q = Distance of the image from the center of the lens

Example 7

Given: The focal length of the lens ($f = 12$ mm) and the distance from the object to the lens ($p = 15$ cm)

Find: The distance from the image to the lens (q)

Solution: Rearrange the equation so that q is isolated. Then substitute in values of other variables and solve for q.

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{f} - \frac{1}{p} = \frac{1}{p} + \frac{1}{q} - \frac{1}{p}$$ (Subtract $\frac{1}{p}$ from both sides.)

$$\frac{1}{f} - \frac{1}{p} = \frac{1}{q}$$ (Subtract out $\frac{1}{p}$'s from right side.)

$$\frac{p}{fp} - \frac{f}{fp} = \frac{1}{q}$$ (Get common denominator on left side.)

$$\frac{p - f}{fp} = \frac{1}{q}$$ (Combine terms on left side.)

We are not there yet; q still must be isolated in the numerator.

$$q\left(\frac{p - f}{fp}\right) = \frac{1}{\cancel{q}} \times \cancel{q}$$ (Multiply both sides by q.)

$$q\left(\frac{\cancel{p - f}}{\cancel{fp}}\right) \times \left(\frac{\cancel{fp}}{\cancel{p - f}}\right) = 1 \times \left(\frac{fp}{p - f}\right)$$ (Multiply both sides by the reciprocal.)

$$* \; q = \frac{fp}{p - f}$$

Before substituting in values of the given variables, make sure all units work together. We were given cm and mm, but we can easily convert mm to cm.

$$\frac{12\ \cancel{\text{mm}}}{} \left| \frac{1\ \text{cm}}{10\ \cancel{\text{mm}}} \right. = 1.2\ \text{cm}$$

$q = \dfrac{(1.2)(15)}{(15 - 1.2)}$ (Substituting in values of variables)

$q = 1.3$ cm (Round to the nearest tenth.)

Therefore, the image is 1.3 cm from the lens.

Before we move on, it is important to note the significance of the derived formula (*). This formula will always find the distance from the image to the lens given the focal length of the lens and the distance from the object to the lens. In general, any formula of the form

$$\frac{1}{A} = \frac{1}{B} + \frac{1}{C}$$

can be rearranged as $C = \dfrac{AB}{B - A}$

or solving for B as $B = \dfrac{AC}{C - A}$

These are useful when finding equivalent capacitance (C_e) of two capacitors (C_1 and C_2) in series because:

$$\frac{1}{C_e} = \frac{1}{C_1} + \frac{1}{C_2} \text{ can now be written } C_1 = \frac{C_e C_2}{C_2 - C_e} \text{ or } C_2 = \frac{C_e C_1}{C_1 - C_e}$$

You may encounter an even more complex equation like this one:

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}$$

If you were asked to solve for s_2 on this problem, the algebra would get quite tedious. So it would be easier to plug in numbers for the given values and then solve for s_2. The next example will demonstrate this idea.

Example 8

Given: A plastic rod ($n_2 = 1.48$) in air ($n_1 = 1$), with one end ground to a convex spherical shape of radius 4 cm ($R = 4$ cm)

$$\frac{n_1}{s_1} + \frac{n_2}{s_2} = \frac{n_2 - n_1}{R}$$

Find: The position of the image (s_2) of a small object 20 cm ($s_1 = 20$ cm) from the spherical end and on axis

Solution: Substitute in given values then solve for s_2.

$$\frac{1}{20} + \frac{1.48}{s_2} = \frac{1.48 - 1}{4}$$ (Substitute in the given values.)

$$\frac{1}{20} - \frac{1}{20} + \frac{1.48}{s_2} = \frac{0.48}{4} - \frac{1}{20}$$ (Subtract $\frac{1}{20}$ from each side.)

$$\frac{1.48}{s_2} = 0.07$$ (Simplify right side with calculator.)

$$\cancel{s_2} \times \frac{1.48}{\cancel{s_2}} = 0.07 \times s_2$$ (Multiply both sides by s_2.)

$$1.48 = 0.07 \times s_2$$

$$\frac{1.48}{0.07} = \frac{\cancel{0.07} \times s_2}{\cancel{0.07}}$$ (Divide both sides by 0.07.)

$$21.1 = s_2$$ (Simplify both sides.)

This solution turns out to be fairly simple. The derivation of the formula for s_2 would have been more difficult than the derivation of the formula for p (Example 6) *and* would have been less useful. So, unless there is a need to use the formula for s_2 a multiple number of times, the derivation is not recommended.

Solution to Scenario Question

Following is the solution to the question posed under "Photonics Scenario Involving Introductory Algebra."

$$G_L = R_1 R_2 G_A^2 (1 - \alpha)$$

$$\frac{G_L}{R_1 R_2 (1 - \alpha)} = G_A^2$$ (Divide each side by $R_1 R_2 (1 - \alpha)$.)

$$\sqrt{\frac{G_L}{R_1 R_2 (1 - \alpha)}} = G_A$$

$$G_A = \sqrt{\frac{0.969}{(0.998)(0.9575)(1 - 0.08)}}$$

$$G_A = \sqrt{1.1022}$$

$G_A = 1.05$, a pure number without units

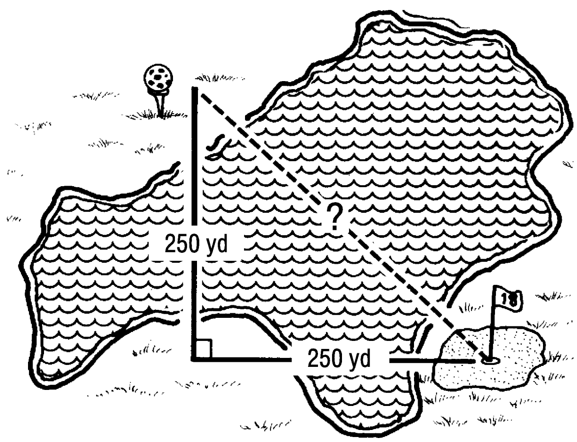
Practice Exercises

Exercise 1

The ratio of the speed of light in a vacuum ($c = 3 \times 10^8$ m/s) to the speed of light in a particular medium (v) results in a constant known as the index of refraction (n). This can also be represented with the formula $n = \frac{c}{v}$. Find the speed of light in glass ($n = 1.5$).

Exercise 2

On a certain golf course, the traditional play is to take two strokes to go around a small lake. Each stroke requires a distance of about 250 yards, as shown below. You wonder how strong a drive would be needed to make it across the lake in one stroke. Determine how far a single stroke would have to go to cross the lake toward the green. (**Note:** For a right triangle, $a^2 + b^2 = c^2$.)



Exercise 3

Your medical insurance policy requires you to pay the first \$100 of your hospital expenses (this is known as a deductible). The insurance company will then pay 80% of the remaining expense. The amount that you must pay can be expressed by the following formula:

$$E = [(T - D) \times (1.00 - P)] + D$$

where E is the expense to you (how much you must pay),
T is the total of the hospitalization bill,
D is the deductible you must first pay,
P is the decimal percentage that the insurance company pays after you meet the deductible.

Suppose you are expecting a short surgical stay in the hospital, for which you estimate the total bill to be about $5000. Use the formula above to estimate the expense to you for the total hospital bill.

Exercise 4

Greenshield's formula can be used to determine the amount of time a traffic light at an intersection should remain green. This formula is shown below.

$$G = 2.1n + 3.7$$

where G is the "green time" in seconds and
n is the average number of vehicles traveling in each lane per light cycle.

Find the green time for a traffic signal on a street that averages 19 vehicles in each lane per cycle.

Exercise 5

White light is incident on the surface of a soap bubble. A portion of the surface reflects green light of wavelength $\lambda_0 = 540$ nm. Assume that the refractive index of the soap film is near that of water, so that $n_f = 1.33$. Estimate the thickness (in nanometers) of the soap bubble surface that appears green in 2nd order ($m = 2$).

Use $2n_f t + \frac{\lambda_0}{2} = m\lambda_0$.

Exercise 6

In close-up photography, the distance of the object from the lens determines how far the lens must be from the film. This requires special lenses and focusing mechanisms. These distances (all measured in the same units) are controlled by the formula.

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

where f is the focal length of the lens,
p is the distance of the object being viewed from the center of the lens, and
q is the distance of the image formed on the film from the center of the lens.

a. Rewrite the formula, isolating the variable representing the object distance.

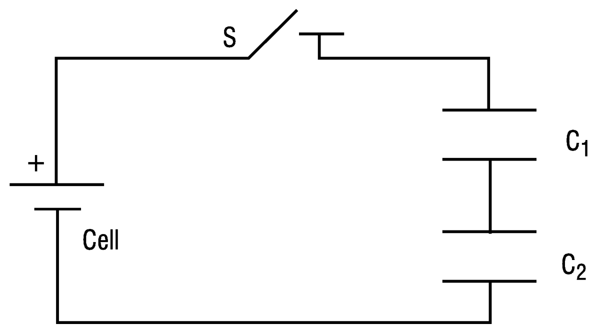
b. Determine the object distance predicted by the equation for a lens with a focal length of 50 mm and a lens to film distance of 6.2 cm.

Exercise 7

Two capacitors are set up in series. Find the equivalent capacitance, C_{eq}, where $C_1 = 2.3\ \mu F$ and $C_2 = 6.5\ \mu F$, using the formula below:

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

(Remember that μ stands for $\times 10^{-6}$.)



Solutions to Practice Exercises

1.

$$n = \frac{c}{v}$$

$$1.5 = \frac{3 \times 10^8}{v}$$

$$v \times 1.5 = \frac{3 \times 10^8}{\not{v}} \times \not{v}$$

$$1.5v = 3 \times 10^8$$

$$\frac{\not{1.5}v}{\not{1.5}} = \frac{3 \times 10^8}{1.5}$$

$$v = 2 \times 10^8 \text{ m/s}$$

The speed of light in glass is 2×10^8 m/s.

2.

$$a^2 + b^2 = c^2$$

$$250^2 + 250^2 = c^2$$

$$125{,}000 = c^2$$

$$\sqrt{125{,}000} = \sqrt{c^2}$$

$$354 = c$$

The hole is about 350 yards away. To clear the water, the ball does not have to go 350 yards in the air, but if it gets over the water it will probably roll that far.

3. $E = \left[(T - D) \times (1.00 - P)\right] + D$

$E = \left[(5000 - 100) \times (1.00 - 0.8)\right] + 100$

$E = (4900) \times (0.2) + 100$

$E = 980 + 100$

$E = 1080$

You will pay $1080 for the hospital bill.

4. $G = 2.1n + 3.7$

$G = 2.1(19) + 3.7$

$G = 39.9 + 3.7$

$G = 43.6$

The light should stay green for about 44 seconds.

5.

$$2n_f t + \frac{\lambda_0}{2} = m\lambda_0$$

$$2n_f t + \frac{\lambda_0}{2} - \frac{\lambda_0}{2} = \mathrm{m}\lambda_0 - \frac{\lambda_0}{2}$$

$$2n_f t = \lambda_0\left(\mathrm{m} - \frac{1}{2}\right)$$ (Factoring out λ_0 simplifies but is not necessary.)

$$\frac{\cancel{2n_f}\, t}{\cancel{2n_f}} = \lambda_0\left(\mathrm{m} - \frac{1}{2}\right) \times \frac{1}{2n_f}$$

$$t = \frac{\lambda_0}{2n_f}\left(\mathrm{m} - \frac{1}{2}\right)$$

$$t = \frac{540}{2(1.33)}\left(2 - \frac{1}{2}\right)$$

$$t = \frac{540}{2.66}(1.5)$$

$$t = 305 \text{ nm}$$

6. **a.**

$$\frac{1}{f} = \frac{1}{p} + \frac{1}{q}$$

$$\frac{1}{f} - \frac{1}{q} = \frac{1}{p} + \frac{1}{q} - \frac{1}{q}$$

$$\frac{q}{fq} - \frac{f}{fq} = \frac{1}{p}$$

$$\frac{q-f}{fq} = \frac{1}{p}$$

$$p \times \left(\frac{q-f}{fq}\right) = \frac{1}{\cancel{p}} \times \cancel{p}$$

$$p \times \left(\frac{\cancel{q-f}}{\cancel{fq}}\right) \times \left(\frac{\cancel{fq}}{\cancel{q-f}}\right) = 1 \times \left(\frac{fq}{q-f}\right)$$

$$p = \frac{fq}{q-f}$$

b.

$$\begin{array}{l|l} 50\ \cancel{\text{mm}} & 1\ \text{cm} \\ \hline & 10\ \cancel{\text{mm}} \end{array} = 5\ \text{cm}$$

$$p = \frac{(5)(6.2)}{6.2 - 5}$$

$$p = \frac{31}{1.2}$$

$$p = 25.8\ \text{cm}$$

7.

$$\frac{1}{C_{eq}} = \frac{1}{C_1} + \frac{1}{C_2}$$

$$\frac{1}{C_{eq}} = \frac{1}{2.3} + \frac{1}{6.5}$$

$$\frac{1}{C_{eq}} = 0.589$$ (using the $1/x$ button on calculator)

$$C_{eq} = 1.7\ \mu\text{F} = 1.7 \times 10^{-6}\ \text{F}$$

Powers and Roots

Objectives

When you have completed this section, you should be able to do the following:

1. Simplify expressions with powers and roots
2. Solve for a variable in an equation with powers and roots

Photonics technicians need to work with expressions involving power and roots, such as the changing intensity I of radiation as a function of distance d from the source, $I = \frac{I_0}{d^2}$, or the threshold gain $(G_A)_{th}$ in a stable laser cavity, where $(G_A)_{th} = \frac{1}{\sqrt{R_1 R_2 L}}$.

Photonics Scenario Involving Powers and Roots

In an optical fiber, different light rays take different times to propagate a given distance through the fiber. A measure of this difference is called the *intermodal dispersion* (τ_i) and is given by the equation:

$$\tau_i = \frac{L}{2n_1 c}(\text{NA})^2$$

τ_i	Intermodal dispersion after propagating a distance L
L	Propagation distance through the fiber
n_1	Index of refraction of the core
c	Speed of light in a vacuum
NA	Numerical aperture equal to $\sqrt{n_1^{\,2} - n_2^{\,2}}$
n_2	Index of refraction of the cladding

Question

A fellow worker asks you to help her calculate the intermodal dispersion for a fiber of length 1 km, where $n_1 = 1.465$, $n_2 = 1.45$ and $c = 3 \times 10^8$ m/s. What value and units do you report for τ_i?

Before looking at the solution, work through the lesson to further develop your skills in this area.

Powers

A power is a short way to write repeated multiplication. The expression 8^4 is a power. The *exponent* 4 represents the number of times the *base* 8 is to be used as a factor.

$$a^n = \underbrace{a \bullet a \bullet a \bullet a \bullet \ldots a}_{n \text{ factors of } a} \qquad \text{Example: } 8^4 = \underbrace{8 \bullet 8 \bullet 8 \bullet 8}_{4 \text{ factors of } 8} = 4096$$

The expression "8^4" is read "eight to the fourth power." Other powers are displayed in the table below.

Exponential Form	Words	Meaning	Value
5^1	5 to the first power	5	5
6^2	6 to the second power, or 6 *squared*	$6 \bullet 6$	36
4^3	4 to the third power, or 4 *cubed*	$4 \bullet 4 \bullet 4$	64
7^4	7 to the fourth power	$7 \bullet 7 \bullet 7 \bullet 7$	2401

Example 1

The *braking distance* is the distance traveled by a car from the instant the driver applies the brakes until the car comes to a stop. The following formula can be used to estimate braking distance.

$$b = \frac{r^2}{30 \times F}$$

where b is the estimated braking distance in feet,
r is the car's speed in miles per hour, and
F is the driving surface factor, given by the following table.

Driving surface factor		
Type surface	**Dry road**	**Wet road**
Asphalt	0.85	0.65
Concrete	0.90	0.60
Gravel	0.65	0.65
Packed snow	0.45	0.45

a. What braking distance would you estimate for a car traveling on dry asphalt at a speed of 55 miles per hour?

b. At the scene of an accident, a car's skid marks indicate that it required about 215 feet to brake to a complete stop on wet asphalt. Isolate r in the formula and find an estimated speed for the car as it began braking.

Solution

a. Use the formula and substitute $r = 55$ and $F = 0.85$.

$$b = \frac{r^2}{30 \times F}$$

$$b = \frac{55^2}{30 \times 0.85}$$

$b = 120$ ft (rounded)

b. Isolate r in the formula. First multiply both sides of the equation by the denominator.

$$b \times (30 \times F) = \frac{r^2}{\cancel{(30 \times F)}} \bullet \cancel{(30 \times F)}$$

Then, to obtain r, find the square root of both sides.

$$r = \sqrt{b \times 30 \times F}$$

Substitute $b = 215$ and $F = 0.65$

$$r = \sqrt{215 \times 30 \times 0.65}$$

$r = 65$ mph (rounded)

Plus and Minus

When finding the power of a *negative* number, we must be especially careful. Recall that

$$5^4 = 5 \times 5 \times 5 \times 5 = 625$$

It may seem simple to also note that

$$(-5)^4 = (-5) \times (-5) \times (-5) \times (-5) = 625$$

Now try it in your calculator and see what result you get. If you remembered the parentheses, you did fine; if not, your answer was negative. It is a good idea to always check you answer with your expectations. Your calculator can mislead you. When you type: -5^4, the calculator essentially reads it as $-(5^4)$. Be sure you understand how the calculator is working before you assume that what it is telling you is true. Raising any number (unless it is zero) to an *even* power will always result in a positive answer.

$b^p = \textit{positive \#}$ where $b \neq 0$ and p is even.

When you have an *odd* exponent, two things can happen. If the base is positive, the result will remain positive; if the base is negative, the result stays negative.

b^p = *positive #* where $b > 0$ and p is odd

b^p = *negative #* where $b < 0$ and p is odd

When you think about them, these rules are rather self-evident.

$4^2 = 16$ and $(-4)^2 = 16$ (Any number raised to an even power is *positive.*)

$4^3 = 64$ (Any positive number raised to any power is *positive.*)

$(-4)^3 = -64$ (Any negative number raised to an odd power is *negative.*)

Roots

Roots are equal factors of numbers expressed as powers.

$$7^2 = 7 \times 7 = 49$$

$$\sqrt{49} = 7$$

The symbol $\sqrt{49}$ asks the question "What number multiplied by itself gives 49?" Or, saying it another way, "What are the two equal factors of 49?"

You can read $\sqrt{49}$ as "the square root of 49." This symbol by itself ($\sqrt{\ }$) always means "the square root."

Your calculator can help you find the roots of numbers. On many calculators, you use the same key to find both the square (x^2) and the square root ($\sqrt{x}$). To use the second meaning of the key, you first press the INVerse key to show that you want the inverse or backward meaning of the key. On some calculators, the INV key may be labeled "2nd F" or just "2nd." This key tells the calculator to use the second meaning of the next key pressed.

Some calculators actually have $\sqrt{x}$ keys.

Example 2

Find the square root of 256.

$\sqrt{256} = ?$

To do this with your calculator,

Enter 256.

(If your calculator requires it) Press the INV (or 2nd) key.

Press the key marked $\sqrt{x}$.

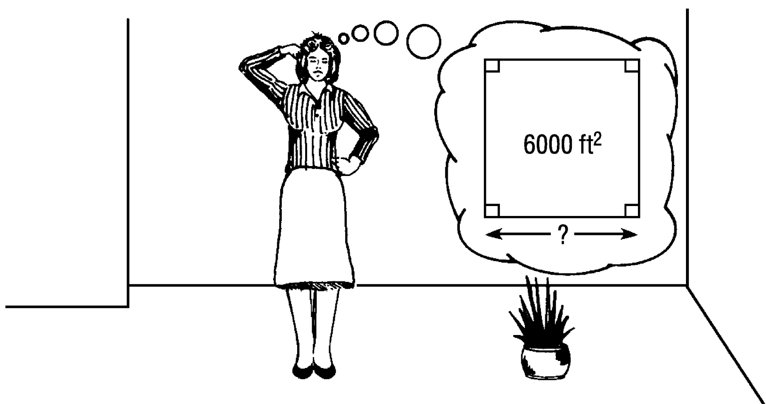
So, $\sqrt{256} = 16$

You can check your answer with the calculator too.

If $\sqrt{256}$ is 16, then 16^2 is 256. Is this true? Use your calculator to find 16^2.

Example 3

The Widget Manufacturing Company needs to build a storeroom that will provide 6000 ft^2 of floor space—as shown below. If the storeroom is going to be square, how many feet on a side will it be?



Square feet of floor space

The formula for the area of a square is

$$A = s^2, \text{ where } s \text{ stands for the side of the square.}$$

Rewrite the formula, substituting what you know for the letters. Since you know the area, the formula now looks like this:

$$6000 \text{ ft}^2 = s^2$$

But you want to find s, not s^2. You should find the square root.

A formula always equates a left side to a right side. To keep the two sides equal, you always do the same thing to both sides.

In this case, you need to find the square root of each side of the formula.

$$\sqrt{6000 \text{ ft}^2} = \sqrt{s^2}$$

Since squaring and finding the square root are inverses of each other, $\sqrt{s^2}$ is just plain s.

What is $\sqrt{6000 \text{ ft}^2}$? To find out, use your calculator. Write the answer with the correct units. Remember, $\sqrt{\text{ft}^2} = \text{ft}$, just as, for example, $\sqrt{4^2} = 4$.

Did you get about 77.46 feet for the side of the square?

Check your answer by squaring it. If you still have the answer in the window of your calculator, just press the x^2 key and you will get back to 6000.

Finding Cube Roots

The root symbol can also be used to mean other roots. You can write a small number near the "bend" of the root symbol to show other roots.

$\sqrt[3]{}$ means "the cube root."

$\sqrt[3]{8}$ asks the question "What are the three equal factors whose product is 8?"

Example 4

What do you think the value of $\sqrt[3]{8}$ is?

$\sqrt[3]{8}$ asks, "If $r \times r \times r = 8$, what number does r stand for?"

Since $2 \times 2 \times 2 = 2^3 = 8$, then $\sqrt[3]{8} = 2$.

Some calculators may have $\sqrt[3]{x}$ keys as well as $\sqrt{x}$ keys.

To find cube roots with a calculator that does not have a $\sqrt[3]{x}$ key, use the y^x key, but this time press the INV key first to get $\sqrt[x]{y}$ or "the xth root of y."

Look at the next example, where a cube root is involved in finding the dimensions of a box.

Example 5

The Acme Storage Company sells a packing box in the shape of a cube that holds 256 cubic inches, as shown below. How long is the edge of the box?

The formula for the volume of a cube is

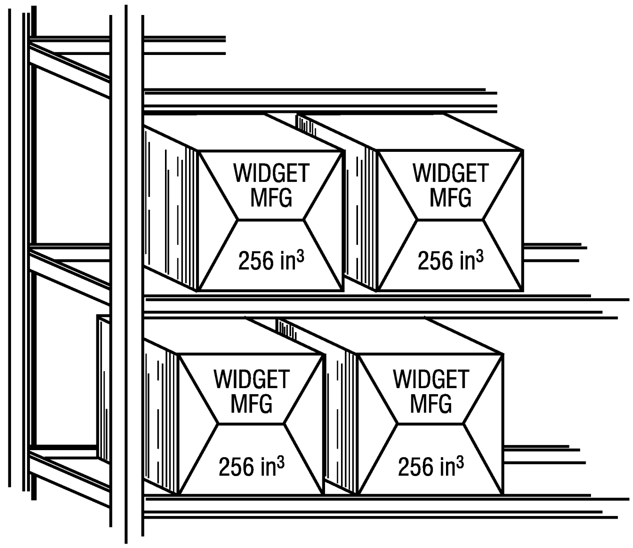
$V = s^3$, where s stands for any one of the edges of the cube.

Rewrite the formula, substituting what you know for the letters. Since you know the volume, the formula now looks like this:

$$256 \text{ in}^3 = s^3$$

To find s, you find the cube root of both sides of the equation.

Since cubing and finding the cube root are inverses of each other, $\sqrt[3]{s^3}$ is just plain *s*.



Packing box

What is $\sqrt[3]{256 \text{ in}^3}$? To find out, use your calculator.

Enter 256.

(Press the INV key or 2nd F key if needed.)

Press the $\sqrt[x]{y}$ key.

Enter 3 (to find the cube root).

Press the = key.

If your calculator has a $\sqrt[3]{}$ key, you need to do only the following:

Enter 256.

(Press the INV key or 2nd F key if needed).

Press the $\sqrt[3]{}$ key.

Write the answer with the units. (Remember, $\sqrt[3]{\text{in}^3}$ = in). Did you get about 6.3 inches for the edge of the cube?

You can use the $\sqrt[x]{y}$ to find many different roots by following these steps:

Enter the number.

(Press the INV key or 2nd F key if needed.)

Press the $\sqrt[x]{y}$ key.

Enter the root that you want.

Press the = key.

Solution to Scenario Question

Following is the solution to the question posed under "Photonics Scenario Involving Powers and Roots."

$$\tau_i = \frac{L}{2n_1c}(\text{NA})^2 = \frac{L}{2n_1c}\left(\sqrt{{n_1}^2 - {n_2}^2}\right)^2$$

$$\tau_i = \frac{1000\,\cancel{m}}{2(1.465)(3 \times 10^8\,\cancel{m}/s)}\left((1.465)^2 - (1.45)^2\right)$$

$$\tau_i = (1.138 \times 10^{-6})(0.0437)$$

$$\tau_i = 4.97 \times 10^{-8}\ \text{s}$$

$$\tau_i = 49.7\ \text{nanoseconds}$$

The intermodal dispersion is about 50 ns per km of fiber.

Practice Exercises

Exercise 1

A pendulum clock can maintain accurate time because it has a finely adjustable period of swing. The period of swing of a pendulum is given by the formula

$$T = 2\pi \times \sqrt{\frac{L}{g}}$$

where *T* is the period (see note below) of the pendulum in seconds,
L is the length of the pendulum in centimeters, and
g is the acceleration due to gravity, 980 cm/s^2.

Note: The **period** is defined as the time required for one complete swing, back and forth.

a. What would be the period of a pendulum that is 25.00 cm long? (Use your calculator's value for π with this formula.)

b. Suppose you want the same pendulum to have a period of 1.000 second. If the length were shortened to 24.90 cm, would the period be closer to or farther from the desired value of 1.000 second?

Exercise 2

You agree to sell your car for \$2600 and allow the buyer to finance it by paying 1% per month interest on the unpaid balance. You would like to amortize the loan in 24 months (finish receiving payment with 24 equal monthly payments). Use the formula below to determine the monthly payment you should ask for.

$$R = P \times \frac{i}{\left[1 - (1 + i)^{-n}\right]}$$

where R is the monthly payment,
P is the loan amount (\$2600),
I is the periodic interest rate (0.01 per month), and
n is the number of payments (periods) (24 payments).

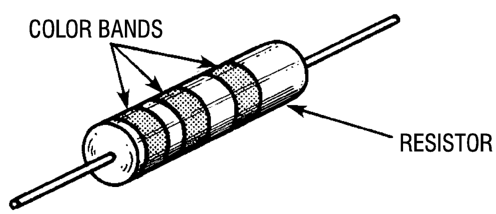
Exercise 3

You want to ship a cube-shaped package that has 8-inch sides. A local vendor sells a cube-shaped shipping box that holds 700 cubic inches. Is this box large enough?

Exercise 4

Electrical resistors are coded with colored bands to indicate the value of resistance in ohms. Each color represents a number:

Color	Number
Black	0
Brown	1
Red	2
Orange	3
Yellow	4
Green	5
Blue	6
Violet	7
Gray	8
White	9



The resistance value is determined by obtaining the first and second digits from the first and second color bands and multiplying by 10 raised to the nth power, where n is the number represented by the third color band. For example, a resistor banded red-brown-orange would have a resistance of 21 (from the red and brown bands) $\times 10^3$ (from the orange band) ohms, or 21,000 ohms.

Determine the resistance values for the following color band combinations:

a. Green-brown-orange

b. Red-red-blue

c. Black-gray-black

d. Yellow-green-brown

Exercise 5

Some infrared temperature detectors use the fact that energy is radiated from hot "black" objects according to the formula

$$R = (5.670 \times 10^{-12}) \times (273°\text{C} + T)^4$$

where R is the total energy radiated in watts per cm² from a perfect radiator
T is the temperature in degrees Celsius

a. What would be the total energy radiated from such an object that had a temperature of 300°C?

b. What would be the total energy radiated from such an object that had a temperature of 37°C?

Exercise 6

When computing statistics, one must often evaluate the expression "n!" or "n factorial." (You may have such a key on your calculator.) Factorial expressions are simply a decreasing series of numbers multiplied together. For example, 4! = 4 × 3 × 2 × 1 = 24. The value of 5! is 120 (5! = 5 × 4 × 3 × 2 × 1). However, manually evaluating factorials of even slightly larger numbers (try 15!) becomes very tedious and results in very large numbers. So, for large values of n, Stirling's formula is often used, as shown below.

$$n! \approx \sqrt{(2\pi n)} \times \left(\frac{n}{e}\right)^n$$

where e is a constant that has a value of about 2.718.

Use Stirling's formula to obtain an approximate value for 30! If your calculator has an x! key, enter 30 and press x! (or 2nd x!) and compare the result with Stirling's formula.

Solutions to Practice Exercises

1. a. $T = 2\pi \times \sqrt{\dfrac{L}{g}}$

$T = 2\pi \times \sqrt{\dfrac{25.00}{980}}$

$T = 1.004$ cm

b. $T = 2\pi \times \sqrt{\dfrac{24.90}{980}}$

$T = 1.002$ cm

The 24.90-cm pendulum is closer to the desired value of 1.000 second.

2. $R = P \times \dfrac{i}{\left[1 - (1 + i)^{-n}\right]}$

$$R = 2600 \times \frac{0.01}{\left[1 - (1 + 0.01)^{-24}\right]}$$

$$R = 2600 \times \frac{0.01}{\left[1 - (1.01)^{-24}\right]}$$

$$R = 2600 \times \frac{0.01}{\left[1 - 0.7876...\right]}$$

$$R = 2600 \times \frac{0.01}{\left[0.2124...\right]}$$

$$R = 2600 \times 0.0471...$$

$$R = 122.391$$

Therefore, the monthly payments will be $122.39.

Note: The "…" in the problem indicates that those numbers are not being rounded, but kept in the calculator. Only at the last step do you round.

3. $8 \times 8 \times 8 = 512 \text{ in}^3$

$512 \text{ in}^3 < 700 \text{ in}^3$

Yes, this box is large enough.

4. **a.** Green-brown-orange

   5-1-3

   $51 \times 10^3\ \Omega = 51\ \text{k}\Omega$

   b. Red-red-blue

   2-2-6

   $22 \times 10^6\ \Omega = 22\ \text{M}\Omega$

   c. Black-gray-black

   0-8-0

   $08 \times 10^0 = 8 \times 1$

   $8\ \Omega$

d. Yellow-green-brown

4-5-1

45×10^1

$450\ \Omega$

5. **a.** $R = (5.670 \times 10^{-12}) \times (273°\text{C} + T)^4$

$R = (5.670 \times 10^{-12}) \times (273 + 300)^4$

$R = (5.670 \times 10^{-12}) \times (573)^4$

$R = (5.670 \times 10^{-12}) \times (1.078 \times 10^{11})$

$R = 0.611\ \text{watts/cm}^2$

b. $R = (5.670 \times 10^{-12}) \times (273 + 37)^4$

$R = (5.670 \times 10^{-12}) \times (310)^4$

$R = (5.670 \times 10^{-12}) \times (9.235 \times 10^9)$

$R = 5.236 \times 10^{-2}\ \text{watts/cm}^2\ (\text{or } 0.05236\ \text{watts/cm}^2)$

6. $n! \approx \sqrt{(2\pi n)} \times \left(\frac{n}{e}\right)^n$

$30! \approx \sqrt{2\pi(30)} \times \left(\frac{30}{2.718}\right)^{30}$

$30! \approx \sqrt{60\pi} \times (11.04...)^{30}$

$30! \approx (13.73...) \times (1.933 \times 10^{31})$

$30! \approx 2.6534 \times 10^{32}$

Entering 30! into a calculator produces a result of 2.6525×10^{32}.

These are extremely close!

RATIO AND PROPORTION

Objectives

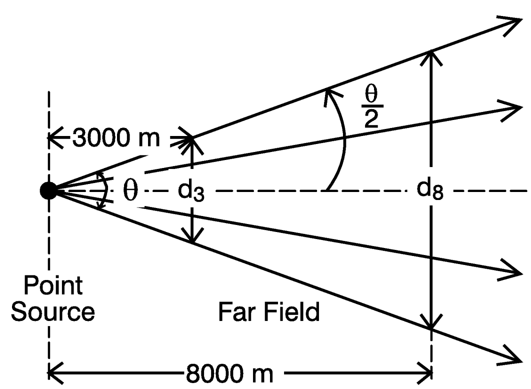
When you have completed this activity, you'll be able to do the following:

1. Understand the concepts of ratio, proportion, and percent
2. Solve problems that involve ratio, proportion, and percent

Photonics technicians need to solve problems that involve ratios, proportions, and percents. For example, they need to interpret drawings scaled at a ratio of 1":400', and solve beam divergence problems where proportions such as $\frac{\sqrt{A_1}}{L_1} = \frac{\sqrt{A_2}}{L_2}$ relate the ratios of cross-sectional beam area to distance along the beam at two positions.

Photonics Scenario Involving Ratio and Proportion

As a technician you have learned that sufficiently far from the output mirror of a laser (in the so-called *far field*), a laser beam appears to radiate from a point source located within the laser cavity. You have just measured the irradiance E in watts/cm² for a CO_2 laser, 3000 meters from the laser, and found it to be 35.6 W/cm². The laser beam spreads as shown below. You figure out that you can set up a proportional variation between the irradiance at 3000 m and that at 8000 m.



Questions

a. What should you expect for the CO_2 laser beam irradiance at 8000 m?

b. If the laser spot is circular and of radius 0.8 meter at the 8000-m position, what is the total power in the CO_2 laser beam?

Before looking at the solutions, work through the lesson to further develop your skills in this area.

What is a ratio?

A ratio is a quantitative comparison of objects and values. Comparisons of one object to another object, and one value to another value, are made in the following statements. Therefore, these statements describe ratios.

1. Bill is twice as old as Jill. The ratio of Bill's age to Jill's age is 2 to 1. If Bill is 2 years old, Jill is 1 year old. If Bill is 10 years old, Jill is 5 years old. Whatever Jill's age is, Bill's age is twice as much. The ratio 2 to 1 can be written several ways. As a fraction, the ratio is written $2/1$. It also can be written as an indicated division ($2 \div 1$) or with a colon (2 : 1). (Incidentally, if Bill is twice as old as Jill is now, it can never happen again. The difference in their ages remains constant, but the ratio of their ages changes each year.)
2. Donuts are being sold by a supermarket at half price. The ratio of this special selling price to the regular price is 1 to 2. If a box of donuts is now sold for $1, its regular selling price is $2. If an individual donut is now sold for 25 cents, its regular selling price is 50 cents. The ratio of 1 to 2 can be written as the fraction $1/2$. It also can be written as $1 \div 2$ or as 1 : 2.
3. A large driven gear has 4 times as many teeth as the 10-tooth drive gear. Here, the ratio of "the number of teeth on the driven gear to teeth on the drive gear" is 40 to 10. This ratio can be written as the fraction 40/10, which can be reduced to $4/1$. The ratio can also be written as $40 \div 10$, or $4 \div 1$. With a colon, the ratio can be written as 40 : 10, or 4 : 1.

The chart that follows sums up the ways in which the ratios in these examples can be written. For each of the examples, the objects were given the same units. Bill's and Jill's ages were both given in years. The price of the donuts was given in units of either dollars or cents. The gears were described by the number of teeth on each.

	In words	**Fraction**	**Indicated division**	**With a colon**
Bill is twice as old as Jill.	2 to 1	$2/1$	$2 \div 1$	2 : 1
They're selling donuts at half price.	1 to 2	$1/2$	$1 \div 2$	1 : 2
The driven gear has 4 times as many teeth as the drive gear.	4 to 1	$4/1$	$4 \div 1$	4 : 1

Sometimes the values to be compared don't have the same units. When this happens, the units for one of the values can usually be changed to the units for the other value. Let's look at an example of how to find the ratio of 15 minutes to 1 hour.

First, write the numbers and units as a fraction: $\frac{15 \text{ minutes}}{1 \text{ hour}}$. Next, either change the units of **minutes** to units of **hours** or change the units of **hours** to units of **minutes**. For this example, we'll change **hours** to **minutes**.

$$\frac{15 \text{ min}}{1 \not{h} \times \frac{60 \text{ min}}{1 \not{h}}} = \frac{15 \not{\text{min}}}{60 \not{\text{min}}} = \frac{1}{4} \qquad \text{(Cancel h and min units.)}$$

Notice that, when the values to be compared both have the same units, the units cancel in the ratio, leaving only a number. Also, the number is here reduced to its lowest terms.

What is a percent?

Now let's look at a type of ratio called **percent**. Percentage is a comparison of **a part** of something to **the whole** of the same thing. The whole is assumed to consist of 100 equal parts.

Percent is just the comparison of a certain number to the number 100. If you have 10 red pencils and 90 yellow pencils, 10 out of the total number of pencils (100) are red. The ratios $^{10}/_{100}$ and 10 : 100 are equal to 10 percent. Since 90 of the pencils are yellow, the ratios $^{90}/_{100}$ and 90 : 100 are equal to 90 percent.

When writing a number as a percent, the symbol "%" is usually used in place of the word **percent**. This means that a value such as 10 percent usually is written as 10%.

Sometimes it's necessary to change a ratio, such as 1 : 6, to a percentage. For example, Figure 2 is a circle that's divided into 6 equal parts. One part is shaded. What percentage of the circle is shaded?

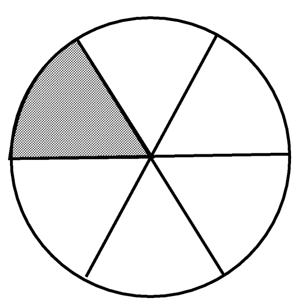


Fig. 2 Circle is divided into 6 equal parts.

Since one part is shaded and there are six parts in the circle, you can write the ratio as $\frac{1}{6}$. To change this ratio to a percentage, divide the number on top (numerator) by the number on the bottom (denominator) to get a decimal number. Then change the decimal number to a "percent" by multiplying by 100. This is easy to do with the help of a calculator. Simply divide the number 1 by the number 6.

Then multiply the calculator answer (0.167) by 100%.

$$0.167 \times 100\% = 16.7\%$$

This answer tells us that 16.7% of the circle is shaded.

What is a proportion?

So far, you've learned that a ratio is simply a comparison of one value to another value. You've also learned about a type of ratio called percent. A percentage compares one part of something to 100 equal parts of the same thing. Now let's learn about one more important type of ratio. It's called a proportion.

Proportion is the relationship of one ratio, such as $\frac{4}{2}$, to another ratio of equal value, such as $\frac{10}{5}$. The phrase "of equal value" is important. For the ratio $\frac{4}{2}$, 4 divided by 2 equals 2. For the ratio $\frac{10}{5}$, 10 divided by 5 equals 2. These ratios have equal value. Therefore, they are said to be proportional.

The ratio $\frac{4}{2}$ is also proportional to $\frac{8}{4}$, $\frac{12}{6}$, $\frac{20}{10}$—and so on. The ratio $\frac{2}{4}$ is proportional to ratios such as $\frac{5}{10}$, $\frac{3}{6}$, and $\frac{8}{16}$ because they all have equal values. If the ratios don't have equal values, they're not proportional.

"Constant of proportionality" is a term used in technology. This term is often shortened to just "constant." As you've seen, when a ratio is proportional to another ratio, the ratios have equal values. Since the values don't change, they're said to be "constant."

For example, a spring is rated by its "spring constant." A spring constant is the ratio of force needed to stretch the spring a certain distance to the distance stretched $\left(\frac{f_1}{d_1}\right)$. It's also equal to the ratio of force needed to stretch the spring another distance to the new distance $\left(\frac{f_2}{d_2}\right)$. These ratios are proportional. Therefore, they're constant.

Let's say that, for a certain spring, a force of 10 pounds is needed to stretch the spring 2 inches and a force of 20 pounds stretches the spring 4 inches. The ratios can be written as:

$$\left[\frac{f_1}{d_1} = \frac{f_2}{d_2}\right] \rightarrow \left[\frac{10 \text{ lb}}{2 \text{ in}} = \frac{20 \text{ lb}}{4 \text{ in}}\right] \rightarrow \left[5\frac{\text{lb}}{\text{in}} = 5\frac{\text{lb}}{\text{in}}\right]$$

The constant value of these ratios is 5 lb/in. Therefore, the spring constant is 5 lb/in. The force needed to stretch or compress a spring, or the distance a spring moves when a known amount of force is applied, can be determined if you know the spring constant.

Solution to the Scenario Questions

Following are the solutions to the questions posed under "Photonics Scenario Involving Ratio and Proportion."

a. The beam spread varies linearly with distance so that

$\tan\frac{\theta}{2} = \frac{\frac{d_3}{2}}{3000} = \frac{\frac{d_8}{2}}{8000}$, so $\frac{d_3}{3000} = \frac{d_8}{8000}$ since the beam-spread angle θ is a constant. The irradiance E varies *inversely* with the cross-sectional area, $\frac{\pi d^2}{4}$, of each target. Thus one can write:

$$\frac{E_{8000}}{E_{3000}} = \frac{A_3}{A_8} = \frac{\frac{\not{\pi} d_3^2}{\not{4}}}{\frac{\not{\pi} d_8^2}{\not{4}}} = \frac{d_3^2}{d_8^2}$$

$$E_{8000} = \left(\frac{d_3}{d_8}\right)^2 E_{3000} \qquad \text{Eqn 1}$$

Since $\frac{d_3}{3000} = \frac{d_8}{8000}$ is a constant ratio,

$$d_3 = \frac{3000 d_8}{8000}$$

So, substituting into Eqn 1 gives us

$$E_{8000} = \left(\frac{\frac{3000 d_8}{8000}}{d_8}\right)^2 (35.6 \text{ W/cm}^2) = \left(\frac{3000 \not{d_8}}{8000 \not{d_8}}\right)^2 (35.6 \text{ W/cm}^2)$$

$$E_{8000} = 5.0 \text{ W/cm}^2$$

b. The power in the beam, P_{beam} = (Irradiance) × (Area)

$$P_{8000} = E_{8000} \times A_8$$

$$P_{8000} = \left(5\frac{\text{W}}{\cancel{\text{cm}^2}}\right)\left(\pi r_8^{\,2}\ \cancel{\text{cm}^2}\right)$$

$$P_{8000} = (5)(\pi)(80)^2\ \text{W} \qquad \text{(0.8 m must be converted to 80 cm.)}$$

$$P_{8000} = 1.0 \times 10^5\ \text{W} = 0.1\ \text{MW}$$

Practice Exercises

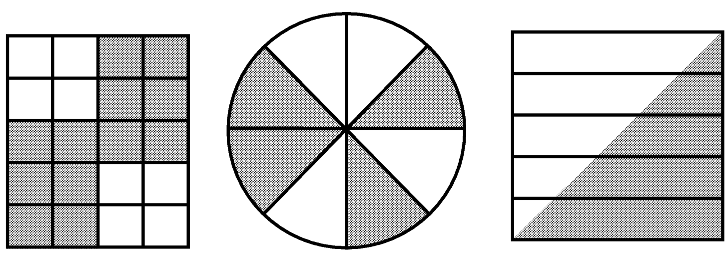
Exercise 1

Read the statements in the following chart. Then write the ratios described by the statements in the blanks. Write each ratio as (1) a fraction, (2) a quotient, and (3) with a colon.

Statement	Fraction	Indicated division	Colon
a. Jana is three times as tall as Mark.			
b. A gallon of milk costs twice as much as a gallon of gasoline.			
c. Her brother is half as old as my brother.			

Exercise 2

What is the ratio of shaded to unshaded areas in the following figures?



a. b. c.

Exercise 3

What **percentage** of each figure in Exercise 2 is **shaded**?

Exercise 4

What **percentage** of each figure in Exercise 2 is **unshaded**?

Exercise 5

Does adding the percentages in Exercises 3 and 4 give 100% for each figure?

Exercise 6

Solder is a mixture of lead and tin. "Soft" solder has 6 parts tin and 4 parts lead.

a. How many grams of tin are in 1 kilogram of soft solder?
(**Hint:** 6 parts + 4 parts = 10 parts = whole.)

b. How many grams of lead are in 1 kilogram of soft solder?

Exercise 7

Which of the following ratios is proportional to $^{40}/_{5}$?

a. $^{20}/_{10}$

b. $^{8}/_{1}$

c. $^{5}/_{40}$

d. $^{80}/_{11}$

Exercise 8

Which of the following ratios is proportional to $^{7}/_{63}$?

a. $^{14}/_{126}$

b. $^{1}/_{8}$

c. $^{1}/_{7}$

d. $^{9}/_{1}$

Exercise 9

What is the constant of the ratios $\frac{12}{1}$ and $\frac{144}{12}$?

Exercise 10

What is the constant of the ratios $\frac{1}{8}$ and $\frac{8}{64}$?

Exercise 11

A force of 70 newtons compresses a spring 2 cm. A second force compresses the same spring only 1 cm. How much force is applied the second time?

(Hint: Use the proportion $\frac{70\text{ N}}{2\text{ cm}} = \frac{x(\text{N})}{1\text{ cm}}$.)

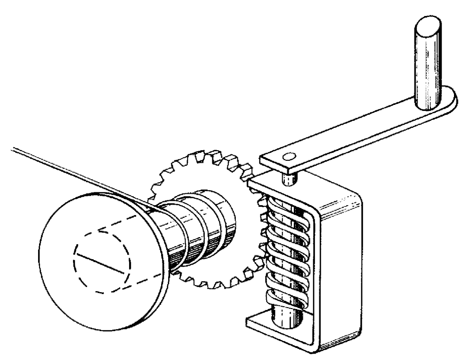
Exercise 12

Blueprints and floor plans of buildings are not drawn actual size but are "scaled down." For example, a 1-foot distance in the building may appear as a ¼" distance on the drawing.

- **a.** What is the ratio that describes the scale used in such a drawing? (Show the ratio both with and without dimensions.)
- **b.** How long a line on a drawing would be used to represent a wall that is 24' long?
- **c.** How long is a duct that is depicted on a drawing by a 9½-inch line?

Exercise 13

A catalog lists the characteristics of several manually operated winches. One of the features listed is the gear ratio. The worm gear shown below is advertised to have a gear ratio of 41 : 1. That is, 41 turns of the hand crank are needed to produce one turn of the large gear.



- **a.** The drum attached to the large gear has a diameter of 1½" and is used to reel in a length of cable. About how many times would the hand crank have to be turned to reel in 18" of cable? (Round to the nearest whole turn of the crank.)
- **b.** Suppose you are able to turn the hand crank at a rate of 40 turns per minute. About how many minutes would it take to wind 18" of cable? (Round to the nearest 0.1 minute.)

Solutions to Practice Exercises

1.

Statement	Fraction	Indicated division	Colon
a. Jana is three times as tall as Mark.	$3/1$	$3 \div 1$	3 : 1
b. A gallon of milk costs twice as much as a gallon of gasoline.	$2/1$	$2 \div 1$	2 : 1
c. Her brother is half as old as my brother.	$1/2$	$1 \div 2$	1 : 2

2. **a.** $\frac{12}{8} = \frac{3}{2}$

b. $\frac{4}{4} = \frac{1}{1}$

c. $\frac{1}{1}$

3. **a.** $\frac{12}{20} = \frac{3}{5} = 0.6 \Rightarrow 60\%$

b. $\frac{4}{8} = \frac{1}{2} = 0.5 \Rightarrow 50\%$

c. $\frac{1}{2} = 0.5 \Rightarrow 50\%$

4. **a.** $\frac{8}{20} = \frac{2}{5} = 0.4 \Rightarrow 40\%$

b. $\frac{4}{8} = \frac{1}{2} = 0.5 \Rightarrow 50\%$

c. $\frac{1}{2} = 0.5 \Rightarrow 50\%$

5. Yes

6. **a.** $\frac{6 \text{ tin}}{10 \text{ total}} \times 1 \text{ kg} = 0.6 \text{ kg}$

$$\frac{0.6\ \cancel{\text{kg}}}{} \left| \frac{1000 \text{ g}}{1\ \cancel{\text{kg}}} \right. = 600 \text{ g of tin in 1 kg of soft solder}$$

b. $\dfrac{4 \text{ lead}}{10 \text{ total}} \times 1 \text{ kg} = 0.4 \text{ kg}$

$$\frac{0.4 \ \cancel{\text{kg}}}{} \ \Big| \ \frac{1000 \text{ g}}{1 \ \cancel{\text{kg}}} = 400 \text{ g of lead in 1 kg of soft solder}$$

7. b.

8. a.

9. 12

10. ⅛

11. $\dfrac{70 \text{ N}}{2 \text{ cm}} = \dfrac{x(\text{N})}{1 \text{ cm}}$

$70 = 2x$ (Cross multiplication)

$x = 35$ N

12. **a.** 1' : ¼"

12" : ¼"

48 : 1 (Multiply both sides by 4 to get whole numbers.)

b. $\dfrac{48}{1} = \dfrac{24'}{x'}$ (Using 48 : 1 ratio)

$48x = 24$

$x = \dfrac{1}{2}' = 6''$

A 24' wall would represent a 6" line on the paper.

c. $\dfrac{48}{1} = \dfrac{x''}{9.5''}$

$x = 456'' = 38'$

A 9½" line on the paper corresponds to 38' in full scale.

13. **a.** Diameter = 1.5" → Circumference = 1.5(π)"

Each turn of the large gear reels in 1.5(π)" of cable. Since 18" is needed, $\dfrac{18}{1.5\pi}$ (which is ≈ 3.82) turns of the *large* gear are required.

$\dfrac{41}{1} = \dfrac{x}{3.82}$

$x = 157$ cranks (rounded to nearest crank)

b. $\dfrac{40 \text{ turns}}{1 \text{ minute}} = \dfrac{157 \text{ turns}}{x}$

$40x = 157$

$x \approx 3.9$

It will take about 3.9 minutes.

EXPONENTS AND LOGARITHMS

Objectives

When you have completed this section, you should be able to do the following:

1. Simplify problems involving natural and common logarithms
2. Convert exponential equations to logarithmic equations

Photonics technicians need to evaluate and simplify expressions and relationships that include exponential and logarithmic functions. Sometime these will involve the common logarithms (base 10), and at other times they will involve the natural logarithms (base *e*). Such expressions are found in the transmission of energy through an absorber of coefficient α in the equation $I = I_0 e^{-\alpha x}$ or the optical density *O.D.* of an eye-goggle filter in the equation $I = I_0 10^{-O.D.}$.

Photonics Scenario Involving Exponents and Logarithms

Questions

a. You plan to use a neutral density filter in a lab setup involving a HeNe laser emitting at 632.8 nanometers. The stamp on the edge of the filter indicates that its optical density (OD) is rated at 0.4 between 400 and 700 nanometers. What percent of the incident HeNe laser light does the filter transmit? Use $T = 10^{-\mathrm{OD}}$.

b. The transmission, *T*, through 5 cm of a certain absorbing material is found to be 60.6%. Find the absorption coefficient of this material. The equation relating the transmission, *T*, thickness of absorber, *t*, and the materials absorption coefficient, α, is: $T = e^{-\alpha t}$.

Before looking at the solutions, work through the lesson to further develop your skills in this area.

The Basics of Exponents and Logarithms

Thus far we have obtained most of the skills necessary to manipulate equations to solve for variables. However, we have yet to consider the following scenario, which requires solving for *x*.

$$3 = 2^x$$

We cannot just move 2 to the other side because *x* is an exponent here. It is necessary to "bring down" the *x*. This becomes possible through what is called a **logarithm**. This allows us to rewrite this equation with the *x* isolated.

$$3 = 2^x \Leftrightarrow x = \log_2 3$$

The above statements are equivalent. In general, we can now write:

$$\underline{\text{Exponential form}} \qquad \underline{\text{Logarithmic form}}$$

$$y = b^x \Leftrightarrow x = \log_b y$$

where *b* is the base, *x* is the exponent, and both *y* and *b* are greater than zero. Look carefully at where each of the three letters (*x, y, b*) goes. Notice that this is a method that solves for the exponent, *x*, so it should be clear that *x* ends up being the isolated variable. It is also worth noting that the letter *b* is the base in both the exponential form and the logarithmic form.

Common and Natural Logarithms

Most of the logarithms you will encounter in photonics will have a base equal to 10 (common log) or the constant *e* (natural log). The common log is typically written "log *x*" and is read "the log of *x*." When no base is indicated, it is assumed to be 10. The natural log is written "ln *x*" and is read "the natural log of *x*." The approximate value of *e* is 2.7183, although your calculator has the better approximation already stored in memory. The use of these logarithms has been helpful in developing relationships (equations) among many measurable characteristics (variables) in optical sciences. For example, the following equation relates the variables for a laser in which the amplifier length has a value of *L* and the mirrors have identical reflectivities *R* and gain coefficient *g*.

$$\left(R \cdot e^{\mathrm{Lg}}\right)^2 = 1$$

This is a rather complex equation, especially when asked to solve for *g*. With a little bit of algebra and log properties, it can be reduced to:

$$g = \frac{1}{2L} \ln \frac{1}{R^2}$$

Most of the work you will do will use formulas such as this one involving logs as shown in the next example.

Example 1

Consider a HeNe laser in which each mirror reflectivity is 99% ($R = 0.99$) and the amplifier length is 20 cm ($L = 20$ cm). Find the gain coefficient, g.

Solution

$$g = \frac{1}{2L}\ln\frac{1}{R^2}$$

$$g = \frac{1}{2(20)}\ln\frac{1}{(0.99)^2}$$

$$g = \frac{1}{40}\ln\frac{1}{0.9801}$$ (Here you must use the "ln" button on your calculator.)

$$g = \frac{1}{40}(0.0201)$$

$$g = 5.03 \times 10^{-4}\ \text{cm}^{-1}$$

Example 2

To ensure that you are able to properly use your calculator for log problems, the following short examples are provided. Check that you can get the same answers on your own.

a. $\log 143 = 2.16$

b. $\ln 65 = 4.17$

c. $5 \log 6 = 3.89$

d. $\frac{2}{3} \ln 8 = 1.39$

e. $\frac{\ln 3}{\ln 2} = 1.58$

f. $\log 10{,}000 = 4$

g. $\log 10 = 1$

h. $\log 1 = 0$

i. $\ln e^2 = 2$

j. $\ln e^3 = 3$

Using log and ln in Algebra

As mentioned earlier $y = b^x$ can be written as $\log_b y = x$. More specifically, write $y = 10^x$ as $\log_{10} y = x$ or even simpler as $x = \log y$. Also $y = e^x$ can be represented with $\log_e y = x$ or better with $x = \ln y$.

Example 3

Solve for x (your answer will include "y").

a. $y = 10^{2x}$ **b.** $y = e^{-3x}$

Solution

a.

$$2x = \log_{10} y$$

$$2x = \log y$$

$$x = \frac{\log y}{2}$$

b.

$$-3x = \log_e y$$

$$-3x = \ln y$$

$$x = -\frac{\ln y}{3}$$

Solution to Scenario Questions

Following are the solutions to the questions posed under "Photonics Scenario Involving Exponents and Logarithms."

a. The equation relating transmission and optical density (OD) is:

$$T = 10^{-OD}$$

$$T = 10^{-0.4}$$

$$T = 0.398$$

The filter will transmit 39.8% of the incident HeNe beam.

b.

$$T = e^{-\alpha t}$$

$$\log_e T = -\alpha t \quad (\text{Since } y = b^x \Leftrightarrow \log_b y = x)$$

$$\ln T = -\alpha t \quad (\text{Because } \log_e x \equiv \ln x)$$

$$\alpha = \frac{-\ln T}{t}$$

$$\alpha = \frac{-\ln (0.606)}{5 \text{ cm}} \qquad \text{(Use ln key on calculator to evaluate.)}$$

$$\alpha = 0.1 \text{ cm}^{-1}$$

Practice Exercises

Exercise 1

Evaluate the following: $\log 10^0$, $\log 10^1$, $\log 10^2$, $\log 10^3$, $\log 10^4$, and $\log 10^5$. Can you make a generalization about $\log 10^n$?

Exercise 2

Evaluate the following: $\ln e^0$, $\ln e^1$, $\ln e^2$, $\ln e^3$, $\ln e^4$, and $\ln e^5$. Can you make a generalization about $\ln e^n$?

Exercise 3

The optical density (*OD*) of a material is the degree of opaqueness it has for a given wavelength of light. The higher the *OD*, the greater the absorption of light. *OD* is used in reference to light filters, laser goggles, and photographic images. For this problem, we will use an equation that defines *OD* in terms of transmittance, *T*. Both *OD* and *T* are unitless variables.

$$OD = -\log T$$

A certain red glass absorbs 99%—and transmits 1% ($T = 0.01$)—of the light entering it when the wavelength of the light is between 400 and 600 nanometers. Find the optical density of the red glass in the 400-to-600-nm range.

Exercise 4

Find the absorption coefficient, a, of a laser propagating at a transmittance of 0.53 through 4.5 cm of the absorbing medium.

$$a = -\frac{1}{x}\ln T$$

where a = Absorption coefficient
x = Distance transmittance is measured into the absorbing medium
T = Transmittance at a distance x into absorbing medium

Exercise 5

Solve for x.

$$y = e^{10x}$$

Solutions to Practice Exercises

1. $\log 10^0 = 0$
$\log 10^1 = 1$
$\log 10^2 = 2$
$\log 10^3 = 3$
$\log 10^4 = 4$
$\log 10^5 = 5$
$\log 10^n = n$

2. $\ln e^0 = 0$

$\ln e^1 = 1$

$\ln e^2 = 2$

$\ln e^3 = 3$

$\ln e^4 = 4$

$\ln e^5 = 5$

$\ln e^n = n$

3. $OD = -\log T$

$OD = -\log (0.01)$

$OD = -(-2)$

$OD = 2$

4. $$a = -\frac{1}{x}\ln T$$

$$a = -\frac{1}{(4.5 \text{ cm})}\ln(0.53)$$

$$a = 0.141 \text{ cm}^{-1}$$

5. $y = e^{10x}$

$10x = \log_e y$

$10x = \ln y$

$$x = \frac{\ln y}{10}$$

GRAPHING IN RECTANGULAR COORDINATES

Objectives

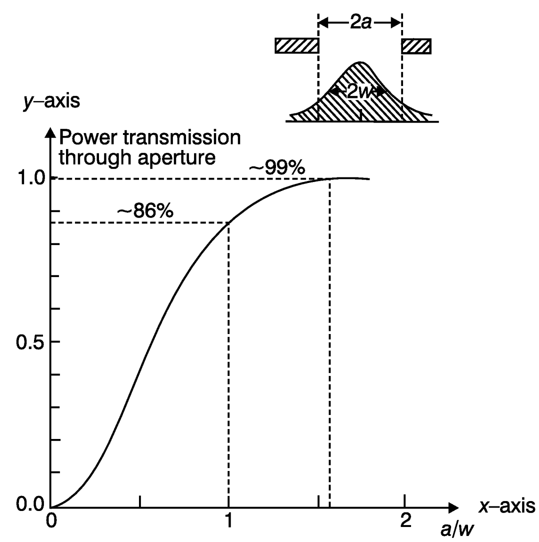
When you have completed this section, you should be able to do the following:

1. Identify ordered pairs in the Cartesian coordinate system
2. Read and interpret line graphs in real-world scenarios
3. Draw graphs from a given table of data

Photonics technicians need to draw, read, and interpret graphs of data and relationships shown on Cartesian coordinate systems. For example, they may need to read traces on an oscilloscope showing voltage V versus time t, or plot the g_1, g_2 cavity parameters on an x-y plot to determine whether or not the cavity is stable.

Photonics Scenario Involving Graphing in Rectangular Coordinates

A TEM_{00} HeNe laser gaussian beam, with spot size, w, is to be directed through a circular opening of diameter, $d = 2a$ (see insert in the upper right corner of the graph). You measure the beam power (in watts) that passes through the circular opening and plot that value against the ratio of the circular opening radius, a, to beam spot size, w, as shown. Power transmission from 0 to 1 (0% to 100%) is plotted along the y-axis. The ratio a/w is plotted along the x-axis. Based on the graph, answer the following questions.



Power transmission of a TEM_{00} cylindrical gaussian beam through a circular aperture, of diameter 2*a*.

Questions

a. How much laser power gets through when the radius, a, of the circular opening is equal to the laser beam spot size, w?

b. For a laser beam of spot size, w, what is the ratio of radius, a, to spot size, w, for a pass-through of 50% of the laser beam power?

c. What is the minimum value of a (expressed in terms of w) for which essentially all of the laser beam power gets through the opening?

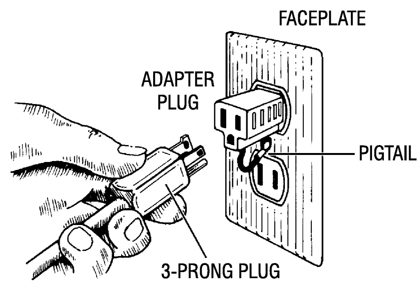
Before looking at the solutions, work through the lesson to further develop your skills in this area.

The Basics of Graphing in Rectangular Coordinates

The figure below explains how to adapt a double-slotted wall outlet to accept a 3-prong plug. The explanation is given in words (part A) and a drawing (part B).

> With a double-slotted wall outlet, use an adapter plug. Connect the green tab or pigtail on the adapter plug by first removing the screw in the center of the receptacle faceplate and then putting the screw through the tab and back into the receptacle. Insert the 3-prong plug into the adapter.

A – Words



B – Picture

Words and a picture

Many people find that part B (the picture) gives them information more easily and faster than part A (the words). Most of us grasp information quickly and easily with a *picture*. Graphs, charts, and tables present numbers and other kinds of information in "picture" form. They provide us with a quicker way to understand the numbers and how they are related.

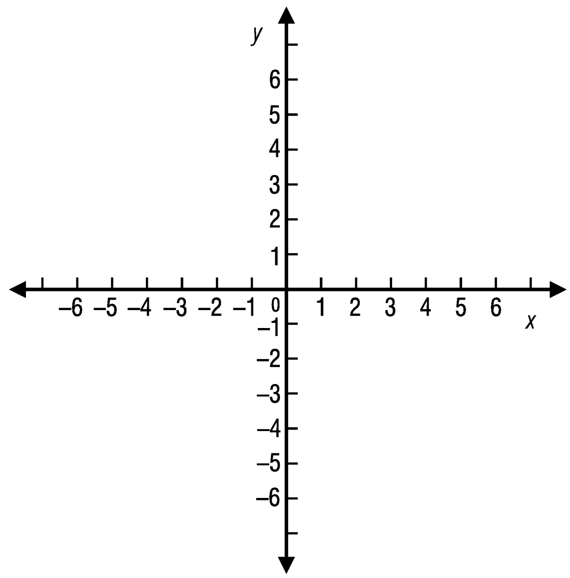
Cartesian Coordinates

The way points and lines are graphed today was first thought of by a French soldier named Descartes over 350 years ago. His idea for picturing points and lines worked so well that the system is named after him—the Cartesian coordinate system.

Example 1

To draw a Cartesian coordinate system, begin by drawing a number line horizontally in the middle of your paper (if you want to use the whole sheet for the graph). This is the x-axis. Place the zero point (or origin) about halfway across the page. Label this point with the numeral 0.

Then draw another number line perpendicular to the first (making a right angle), crossing the first line at the origin (or zero point). This is the y-axis. Notice that the positive numbers are to the *right* of the origin for the horizontal line and *up* from the origin for the vertical line.



The Cartesian coordinate system

The size of the unit (the distance from zero to one) does not *have* to be the same on both number lines. But it is generally a good idea to draw the units the same unless you have a special reason for making them different. For example, you might need to have many small units on one line and only a few large units on the other.

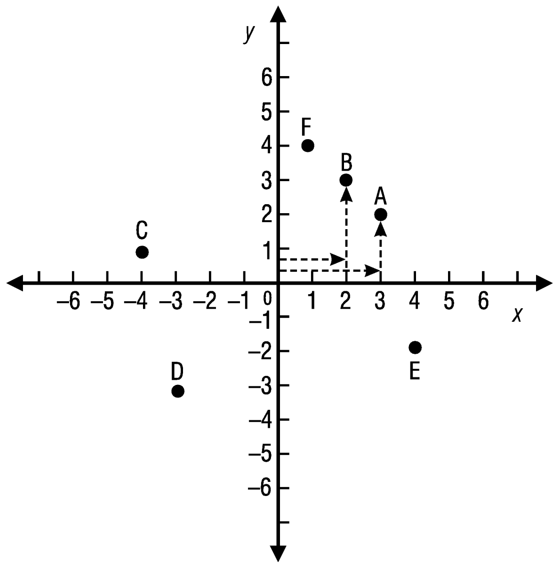
However, once you draw a unit from zero to one on each of the number lines, the size of the unit is always the same on the positive and negative halves of the number lines.

The horizontal number line and the vertical number line are both called axes. The horizontal axis is perpendicular to the vertical axis and they cross at the origin, or zero point.

Every point on the Cartesian coordinate system can be named by a pair of numbers, or coordinates. The *first coordinate* always tells how *far to the right* or *left* of the origin the point is. The *second coordinate* in the pair always tells *how far up or down* from the origin the point is. The number pair that names the origin is (0, 0). This is why it is often called the zero point.

Example 2

Write the number pairs that identify the points A, B, C, D, E, and F.



To find the number pair that identifies point A, begin at the origin where the axes cross. Move to the *right* along the horizontal axis until you are under point A. As you move, *count how many units to the right* you move from the origin. Write down this number.

Then move *up* along a line parallel to the vertical axis until you reach point A. As you move, count *how many units up* you move from the x-axis. Write this number to the right of the first number, place a comma between them, and enclose the pair of numbers in parentheses.

Did you get (3, 2) as the pair of numbers that identifies point A?

To find the pair of numbers that identifies point B, start again at the origin and follow the same process. Did you get (2, 3)?

Does it make any difference which number you write first? Yes, the order of the numbers does make a difference. Point A is (3, 2) and point B is (2, 3)—and they indeed are not the same point.

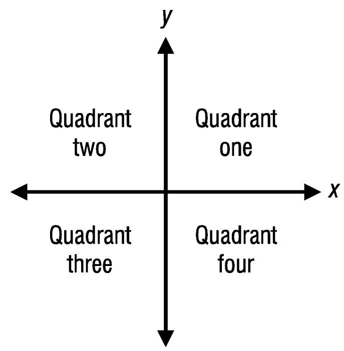
The pair of numbers that identifies a point in the Cartesian coordinate system is called an **ordered pair** because the order of the numbers makes a difference.

The horizontal axis is called the x-axis, and the vertical axis is called the *y-axis*. The *ordered pair* of numbers that identifies any point is (x, y). The x-value of a point is always written first; the y-value is always written second.

The coordinates of the remaining four points are as follows: C (–4, 1), D (–3, –3), E (4, –2), and F (1, 4). Remember: If the point lies to the *left* of the y-axis, its x-value will be negative. If the point lies *below* the x-axis, its y-value will be negative.

Quadrants

The Cartesian coordinate system divides flat space like your paper into four quadrants. The figure below shows how these quadrants are numbered.



Naming the quadrants

Example 3

Use the figure above and the ordered pairs that you wrote for Example 2 to help you choose the correct word (*positive* or *negative*) to complete each of these statements.

a. All the points in the first quadrant have x-values that are _______ and y-values that are _______.

b. All the points in the second quadrant have x-values that are _______ and y-values that are _______.

c. All the points in the third quadrant have x-values that are _______ and y-values that are _______.

d. All the points in the fourth quadrant have x-values that are _______ and y-values that are _______.

Solution

a. positive, positive

b. negative, positive

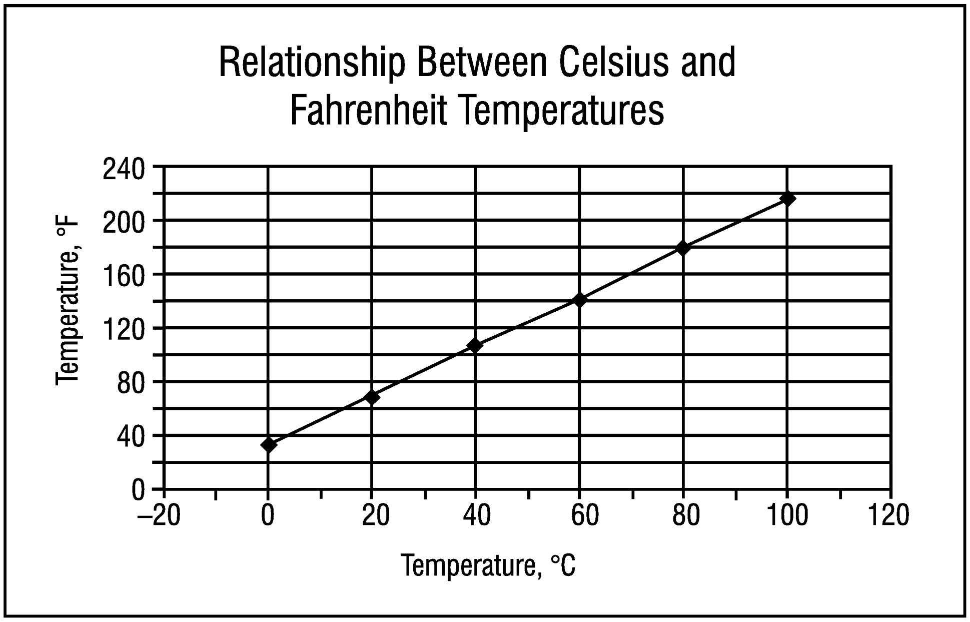
c. negative, negative

d. positive, negative

Reading Graphs

In the Cartesian coordinate system, sometimes points can be connected in a line, sometimes in a curve. Either way, it is important that you understand not just how to plot points, but also how to interpret the graphs they create.

You may have seen thermometers that are labeled with both Celsius and Fahrenheit temperature scales. On these thermometers, you can read the temperature in degrees Fahrenheit or in degrees Celsius. But what if your thermometer has only one temperature scale and you need the other? What if you want to know how the Fahrenheit and Celsius temperature scales are related? You might look for a table that has a matching list of Fahrenheit and Celsius temperatures. Or you might use a conversion formula and a calculator to calculate the value you need. Another way is to use a line graph. The figure below shows a line graph that relates Celsius and Fahrenheit temperatures.



To figure out what the graph contains, first look at the labels. What temperature scale is shown on the bottom axis? What temperature scale is shown on the side axis?

You can see in the graph that the line slopes up, from left to right. This means that for low Celsius temperatures you can expect low Fahrenheit temperatures—and for high Celsius temperatures you can expect correspondingly higher Fahrenheit temperatures. The graph is like a table of data. Each dot on the line represents equal Celsius and Fahrenheit temperatures. Let's continue to look at the graph to figure out some equivalent temperatures.

Example 4

What Fahrenheit temperature corresponds to 20°C?

Look at the bottom axis of the graph. Find the place on the axis that represents 20°C. From here, go straight up until you reach the graphed line. There should be a dot there. From that dot go straight left to a location on the side axis. What is the Fahrenheit value represented by this point on the side axis? The point on the side axis may not be a labeled point. In this case, you need to estimate the value, using the adjacent temperature values that are shown. What do you get? Do you get about 70°F? (The exact value is 68°F.)

Example 5

What Celsius temperature corresponds to 32°F?

You can also read a line graph the "other way." Look at the vertical axis of the graph. Find the place on the axis that represents 32°F. From here go straight to the right until you reach the line graph. There should be a dot there. From the dot on the line graph, go straight down until you reach a point on the bottom axis—the corresponding Celsius temperature. What is the Celsius value at this point on the axis? Do you get a value of 0°C? Is that what you expected?

What if you want to read a temperature that is not shown with a dot? That's when the line drawn through the dots can help you. The line shows that the dots follow a trend; if more dots were plotted, they would all be on this line. Whenever you use the line to estimate values between the plotted values (dots), you are *interpolating* between the data. Let's interpolate between the graphed data to find a temperature not shown by a dot.

Example 6

What Fahrenheit temperature corresponds to 50°C?

As before, find the place on the horizontal axis that corresponds to 50°C. Then go straight up until you meet the graphed line. This time you don't find a plotted point—or dot—on the line. However, since the line is drawn, you can *assume* that a dot is there. From this point, go straight to the left, as before, until you reach the side axis. What Fahrenheit temperature do you find? Do you get a little bit more than 120°F?

Sometimes you need values that are beyond those shown on a graph. You would like to be able to extend the graph, relying on the trend shown by the graph. This is called *extrapolating*, or extending the graph. Extrapolating normally requires some understanding of how the data on the graph are related. In the case of Fahrenheit and Celsius temperatures, you probably already know that they are related by a simple linear formula. So you know that the relationship or trend shown will continue for higher and higher Celsius and Fahrenheit temperatures (as well as for lower and lower ones). You could sketch in what you expect the graph to look like by extending the line that is already drawn.

Example 7

Find the Fahrenheit temperature that is equal to a temperature of 110°C.

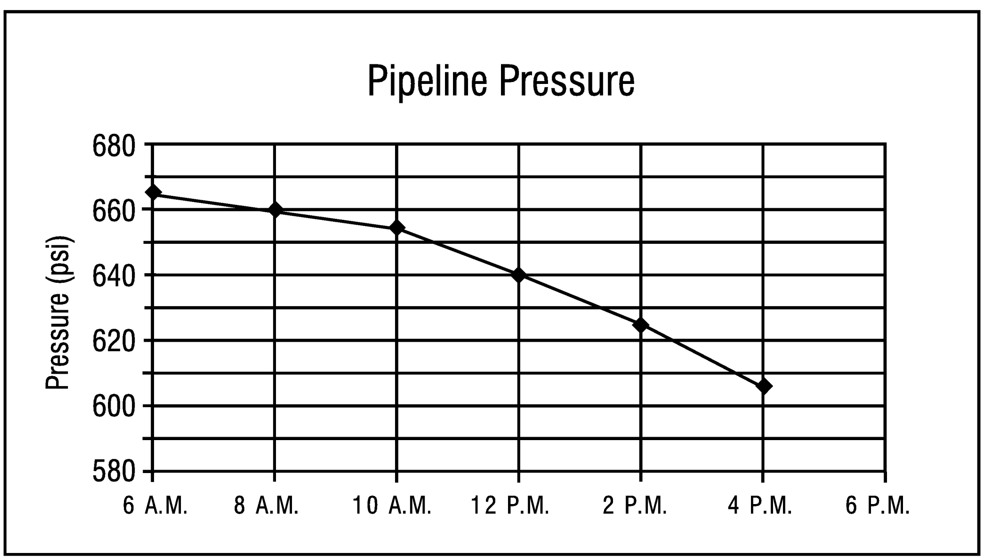
On a worksheet, trace the graph, leaving room on the right and above. Now use your ruler to extend the line graph on your paper beyond the value corresponding to 110°C on the horizontal axis. Then, using the procedure for reading the graph, find the temperature in degrees Fahrenheit that equals a temperature of 110°C. Do you get about 230°F?

In the graph, the horizontal and vertical axes were already labeled out far enough to allow you to extend the graphed line. Sometimes, you will have to extend these axes, too, so you can properly draw the rest of the graph and read it.

Now let's consider an example of a line graph on which the relationship is better pictured by a curved line rather than a straight line.

Example 8

The graph below shows a line graph of how the pressure in a high-pressure pipeline changes from morning to afternoon on a certain day. Notice the label on the side of the graph. It tells you the values of pressure in "pounds per square inch" (psi) that are plotted on the graph. The data are plotted with dots and joined with a line. Notice that the line is a curved line. When you look at this graph, you can compare the afternoon value with the morning value and see that the pipeline pressure is getting lower.



You can also see that the pressure is dropping faster in the afternoon than in the morning. When you look at a line graph and notice how the values are changing, you are observing a *trend* or *pattern*. This is probably the most useful feature of line graphs, particularly graphs that use "time" on the bottom axis.

You can read values from the curved line in the graph, just as you did with the temperature-conversion graph. You may wonder what happened to the pressure between 10 A.M. and 12 P.M. The plotted points tell what the pressure was at 10 A.M. and at 12 P.M. Can you determine what the pressure was at 11 A.M.? As before, you must interpolate to find the answer. The curved line indicates that the pressure at 11 A.M. was somewhere between the values at 10 A.M. and 12 P.M., perhaps 648 psi. Does that seem reasonable?

Look again at the graph. Suppose you wanted to know what the pressure was going to be at 6 P.M. Can you extend this graph out to 6 P.M.? How confident would you feel about the value you determined by extending the graph?

Drawing Graphs

You will probably have many occasions to draw line graphs. Follow these steps to draw a line graph:

First, decide what you are trying to show. Choose a title that describes your graph accurately.

Second, choose the general labels for the axes. It may be helpful to arrange your information in a table first, to help you see clearly what you want to graph.

Third, figure out what units you want for your graph. Draw your axes. Mark and label them with the units you've selected.

Fourth, by going over and up from the origin, plot the points for your graph.

Fifth, begin at your first point and draw a line through the remaining points that best represents the data you've drawn. The line may be straight or curved, depending on the relationship among the values you are graphing.

Ohm's law states that in a circuit the product of the current (I) and resistance (R) is equal to the applied voltage (V) or $V = I \times R$. Suppose currents and voltages were found as seen in the table below.

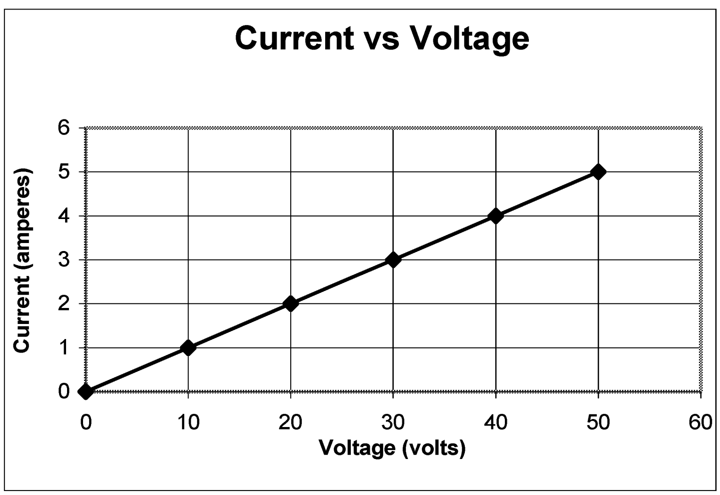
Voltage (Volts)	Current (Amperes)
0	0
10	1
20	2
30	3
40	4
50	5

Look at the table of numbers you are going to plot to help you decide what title and labels to use for this graph. Next, draw the axes on a sheet of graph paper. Using regular spacing along the axes (you should be able to use the lines printed on the graph paper), mark the voltage along the x-axis (bottom); it is the *independent variable*. Then current goes on the y-axis (left) because it is the *dependent variable* (varies according to the independent variable). Another way to think of these terms is that the independent variable is the cause while the dependent variable is the effect.

Now you are ready to plot the points in the table. To plot the first point in the table (0 volts giving 0 amperes), begin at the origin and go 0 units over and then 0 units up. Of course you don't go anywhere! The origin is the first point on your line graph. Make a small dot at the origin to show that you have graphed these values as the first point.

To plot the second point in the table (10 volts giving 1 ampere), begin at the origin and go over to 10 along the horizontal axis. Then from there go 1 unit straight up. Make a small dot at that point. In the same way, plot the third, fourth, fifth, and sixth sets of data.

Now use a straightedge to draw a line that joins the first point to the second point, the second point to the third, and so on. This particular graph is a straight line. Is the line you drew a straight line (or fairly close)? If it isn't, plot the points carefully again. Compare your graph to the one shown below and make any needed corrections.



You may have noticed that we have not mentioned the third variable of Ohm's law, resistance. Solving for resistance in Ohm's law would result in $R = \frac{V}{I}$. So, for any ordered pair on the graph (or table), we can find the resistance by dividing the voltage by the current (or the variable on the x-axis divided by the variable on the y-axis).

$$R = \frac{V}{I} = \frac{\text{voltage}}{\text{current}} = \frac{x}{y}$$

You may also recognize that $\frac{x}{y}$ is essentially the inverse of the slope $\left(\frac{\Delta y}{\Delta x}\right)$. Therefore, in this problem the resistance (R) is the inverse of the slope (m) of the graph.

$$m = \frac{\Delta y}{\Delta x} = \frac{1}{R}$$

Solutions to Scenario Questions

Following are the solutions to the questions posed under "Photonics Scenario Involving Graphing in Rectangular Coordinates."

a. When $a = w$, the ratio of $a/w = 1$. Locate the point where $a/w = 1$ on the x-axis (same as the value $d = 2a = 2w$). Next move vertically up the $a/w = 1$ line to the curve and then horizontally over to the y-axis. You will see that about 86% of the laser power gets through.

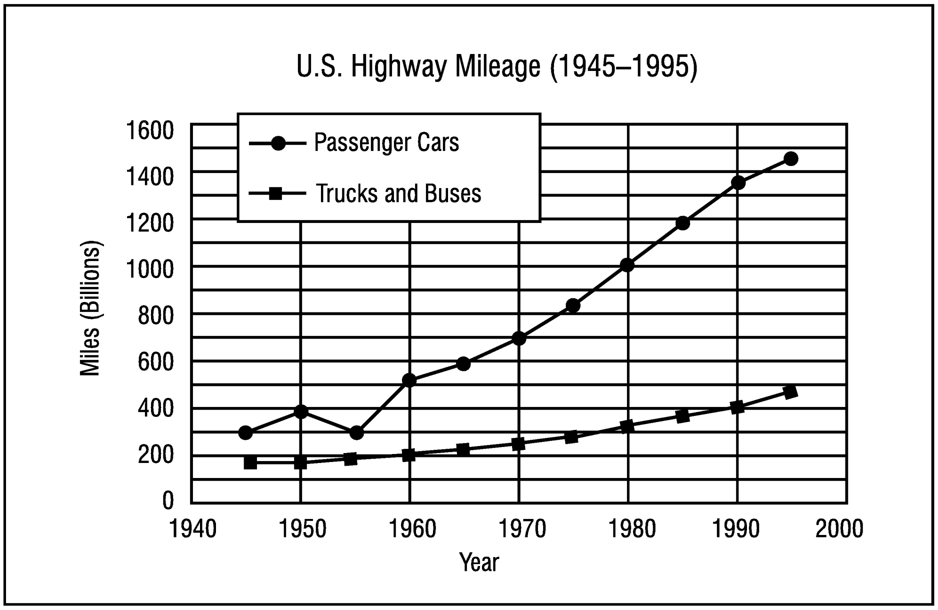
b. Starting at power = 50% (the 0.5 point on the y-axis), move horizontally across to the curve and vertically down to the x-axis. You will find that a/w is approximately equal to 0.6.

c. For 100% power transmission, the ratio a/w is about 1.6. So the minimum value of radius a is 1.6 times the laser beam spot size (1.6 w).

Practice Exercises

Exercise 1

Information about the total highway mileage traveled by vehicles in the United States is shown below.



a. What types of vehicles are referred to by this graph?

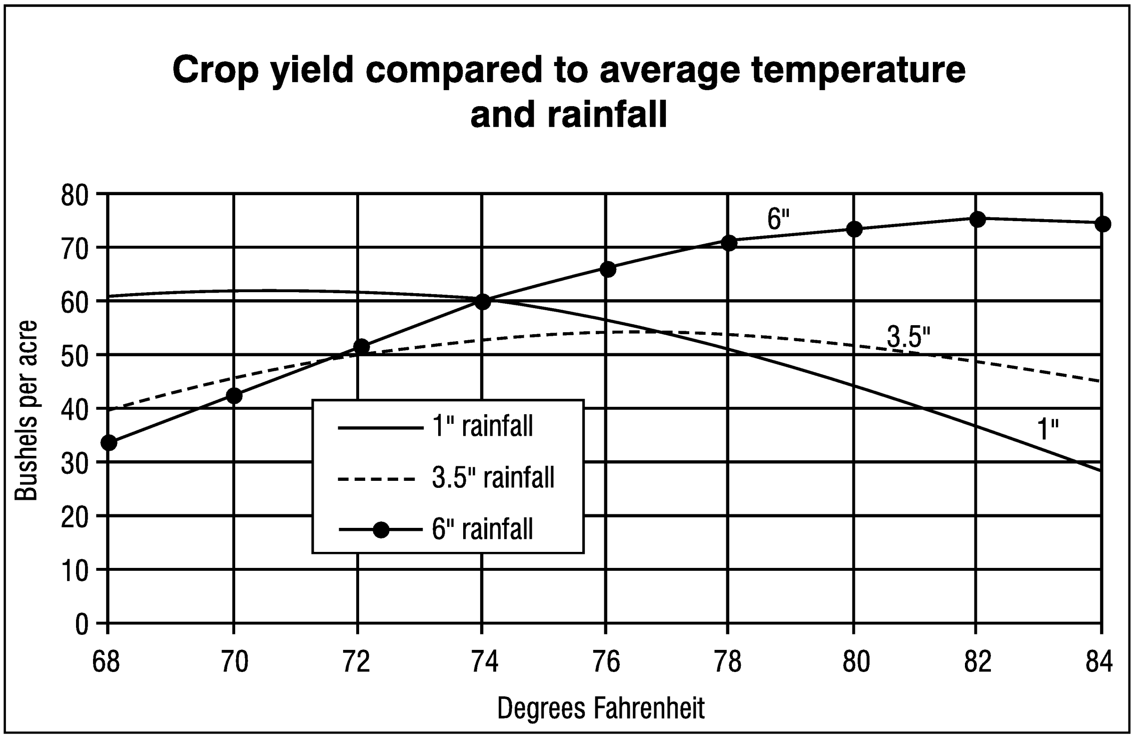
b. What span of time is covered by the graph?

c. The total mileage traveled each year is increasing. Which type of vehicle is increasing its total mileage faster?

d. Use the graph to estimate how many miles trucks and buses will travel in the year 2000. Do you think this would be a reliable estimate? Why?

Exercise 2

There are many relationships between weather and crop performance.
A relationship pertaining to corn yields in a certain region is pictured below. It compares crop yield with average temperature for the month of July, for various rainfalls.



a. What units are being used for (1) crop yields, (2) average temperature, and (3) rainfall?

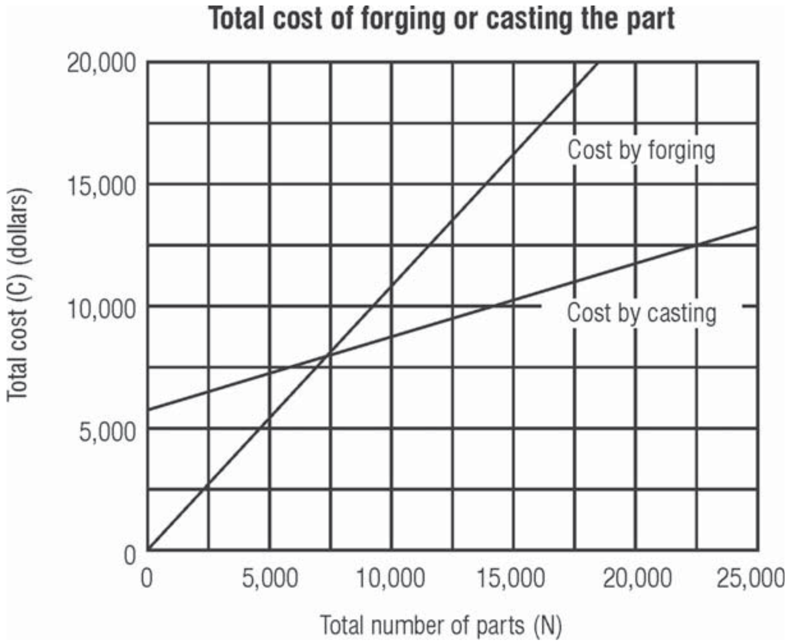
b. How is the graph drawn so that you can identify the lines for the three different rainfalls?

c. For an average temperature of 82°F, what crop yields can be expected for a 6" monthly rainfall? For a 1" monthly rainfall?

d. Which is better for a cool average temperature of 70°F—heavy rainfall or light rainfall?

Exercise 3

A large company produces a certain machine part. The company can produce the part either by casting or by forging. The tooling cost for casting is quite expensive, but the labor and material cost is relatively low compared to the forging method. With the forging method, the tooling cost is lower but the labor and material cost is higher. A graph of these relationships is as follows.



a. For an order of 15,000 parts, what is the cost of producing the part by casting? By forging?

b. For an order of 5000 parts, what is the cost of producing the part by casting? By forging?

c. At what size order, the breakpoint, are the cost of forging and the cost of casting the same?

d. Can the "breakpoint" determined above be used as a guideline for the production staff? What would that guideline be?

Exercise 4

Traveling at 50 miles per hour, you record the mileage on your odometer every 30 minutes, as shown below.

Time (min)	Distance traveled (mi)
0	0
30	25
60	50
90	75
120	100

a. Draw and label the axes for your graph. Use the labels and title in the table above to help you label the axes. Put the driving time along the x-axis (0, 30, 60, 90, and 120) and distances traveled along the y-axis (up to 100). Then plot the information from the table, using the driving time and the distance traveled. Plot each set of data given in the table and join the points on the graph with lines. Your graph should be a straight line. If your graph is not fairly straight, check your points and plot them again.

b. According to your graph, on your next trip (at the same speed) about how many miles can you travel in 50 minutes? How many in 100 minutes? Remember that you are interpolating with your graph; the result is only an estimate.

c. Suppose you wanted to show the bottom axis in hours, rather than minutes. Relabel your bottom axis in hours without replotting your points.

d. Now use your graph to tell how far you could travel in 3 hours. Do you think this is a reasonable extrapolation of your graph?

Exercise 5

As a lab technician, you must perform various tests of material properties. A stress analysis on wire samples is one such test. Shown below are the results of a stress analysis on a length of copper wire. The stretch of the wire is measured as different weights are hung from it.

Stress analysis on copper wire

Load (lb)	Stretch (inches)
0	0.00
2	0.02
4	0.04
6	0.06
8	0.08
10	0.10
12	0.13
14	0.64

a. Construct a graph of the stretch in the copper wire at each load. Label the load along one axis and the stretch along the other axis.

b. Interpret the graph to estimate the load limit for this wire. Show the load at which the wire will stretch to the point of breaking.

c. Can you estimate what the stretch might be at 3 pounds? At 16 pounds? If so, what is your estimate? If not, explain why you are not able to estimate.

Solutions to Practice Exercises

1. **a.** Passenger cars and trucks and buses

   b. 1945 to 1995

   c. Passenger cars

   d. About 550. This estimate should be fairly reliable because over the last 50 years the increase has been consistent.

2. **a.** (1) bushels per acre, (2) Fahrenheit, (3) inches

   b. Each line is formatted differently so the user can distinguish between them.

   c. 6" about 75 bushels per acre

   1" about 37 bushels per acre

   d. Light rainfall (1" rainfall results in about 62 bushels per acre.)

3.

a.	15,000 parts	casting	about \$10,500
		forging	about \$16,000
b.	5000 parts	casting	about \$7000
		forging	about \$5500

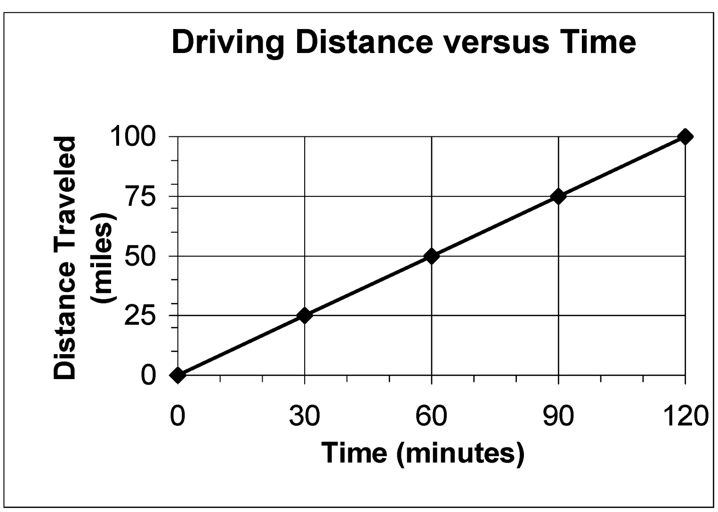
   c. About 7500 parts

   d. Yes. If producing less than 7500 parts, it is more cost effective to use the forging method. If over 7500 parts are being produced, casting will save money.

   Parts < 7500 → forging

   Parts > 7500 → casting

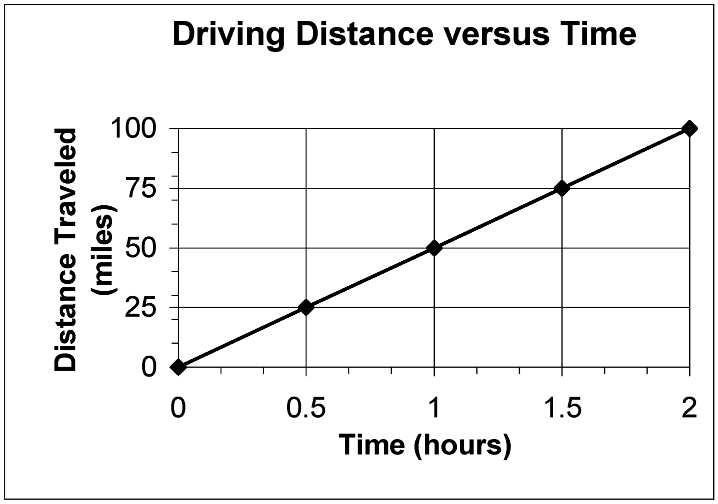
4. **a.**



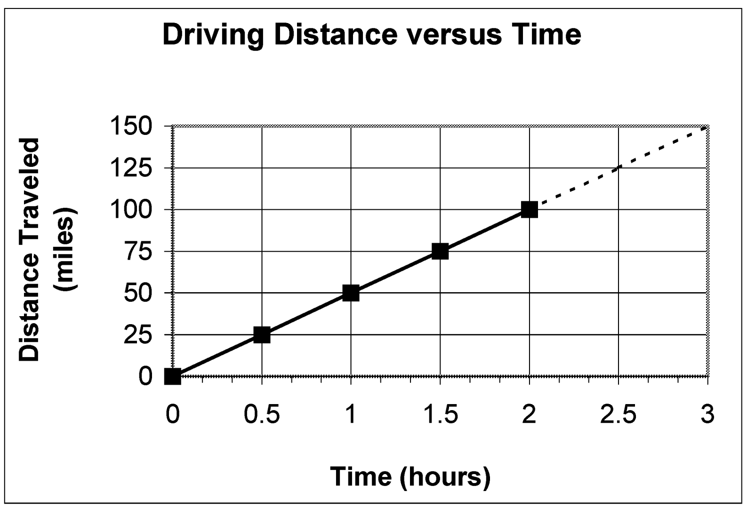
b. In 50 minutes you would travel about 42 miles.

In 100 minutes you would travel about 84 miles.

c.

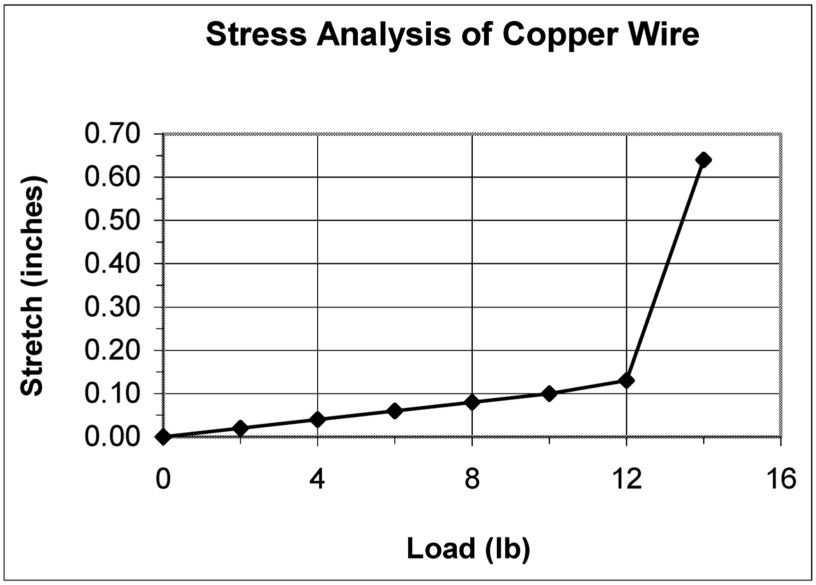


d.



150 miles; this is a reasonable extrapolation. However, the more time spent in the car, the greater the likelihood that you will stop for gas, food, etc.

5. **a.**



b. Somewhere between 12 and 14 pounds the stretch distance increases drastically.

c. 3 pounds → about 0.03 inch

At 16 pounds an accurate estimation is not possible. We know that it would exceed 0.64 inch, but we know little beyond that. The problem is that there is not a consistent pattern at and around this point to project the stretched distance. The "point of breaking" in a material can be modeled with a piece of silly putty. Take a piece and pack it tightly together. Then slowly pull on both ends. You'll notice that initially there is resistance to the stretching. But once it gets to a certain point the resistance greatly decreases as you pull it apart.

GEOMETRY

Objectives

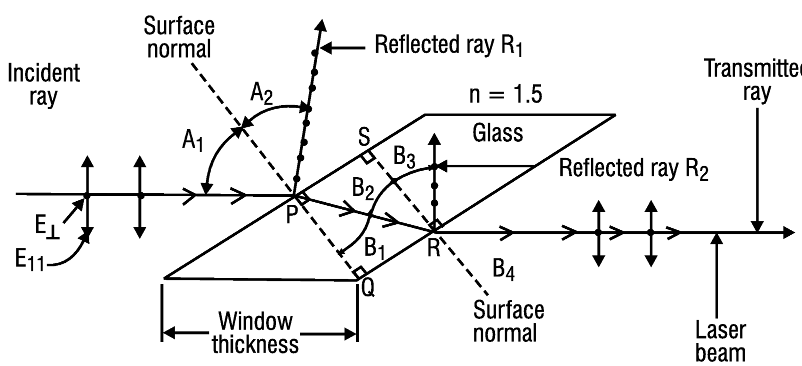
When you have completed this activity, you should be able to do the following:

1. Recognize parallel and perpendicular lines and understand their properties
2. Measure angles
3. Find unknown angle measures in triangles algebraically

Photonics technicians need to use the geometrical properties of parallel and perpendicular lines to solve problems. They also must measure angles between lines that are not perpendicular or parallel, and use the properties of triangles to solve for angles that cannot be easily measured. These skills are used to calculate Brewster's angle of reflection for producing polarized light or Snell's Law of Refraction for tracing light rays through an interface separating two different optical media.

Photonics Scenario Involving Geometry

Brewster's windows are used in laser cavities to achieve polarization of the laser beam. The drawing below shows an unpolarized incident ray of laser light incident at angle A_1 on the left surface of a window of index of refraction $n = 1.5$. The ray reflects (ray R_1) and refracts (ray PR) at the left surface. Ray PR goes through the glass and reflects (ray R_2) and refracts at R on the right window surface. The transmitted laser beam is now partially polarized.



Questions

Based on the drawing, determine the following:

a. Angle A_1 if it is to be equal to Brewster's angle B where $\tan B_1 = n$

b. Angle A_2 for the reflected beam R_1

c. Angle of refraction B_1 for angle A_1 equal to Brewster's angle B

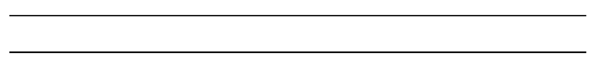
d. Angles B_2 and B_3 in terms of refracted angle B_1

e. Angle of refraction B_4 at the right window surface for incident angle B_2.

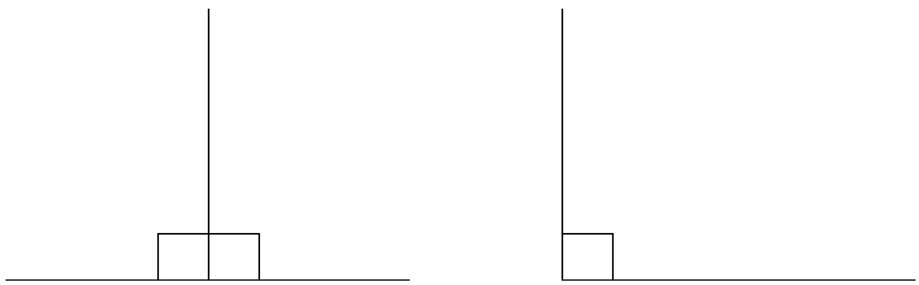
Before looking at the solutions, work through the lesson to further develop your skills in this area.

Lines

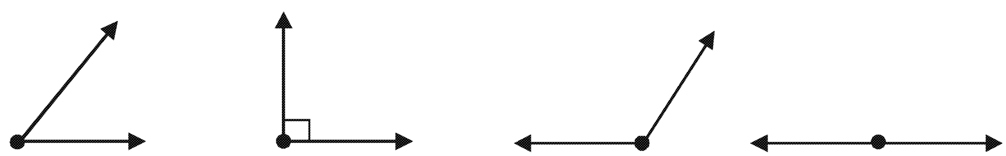
Parallel lines—Lines in the same plane that never intersect are parallel lines.



Perpendicular lines—Lines that intersect to form a right angle are perpendicular lines. Note the "square symbol" is used to identify a right angle.



Angles—An angle is the figure formed by two rays with a common endpoint. Angles may be classified by their measures.



Acute Angle An angle whose measure is greater than 0° and less than 90°

Right Angle An angle whose measure is 90°

Obtuse Angle An angle whose measure is greater than 90° and less than 180°

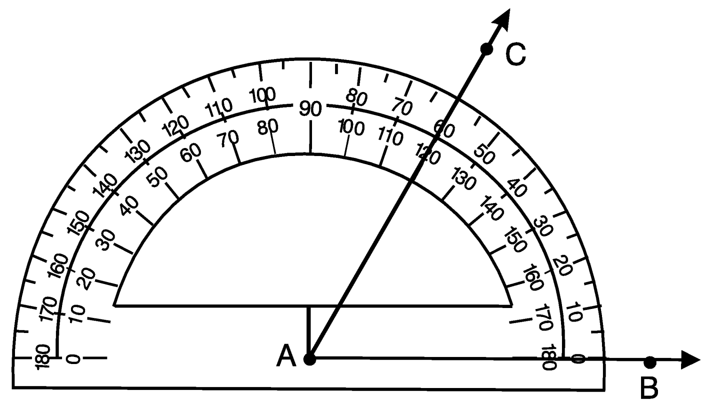
Straight Angle An angle whose measure is 180°

Angles are measured as parts of a circle. For measuring in degrees, the circle is divided into 360 parts. Each part is one degree, written 1°.

A **protractor** is a tool for measuring angles.

Example 1

Measure ∠CAB in the illustration below.



Solution

a. Place the midpoint mark of the protractor over vertex A of ∠CAB.

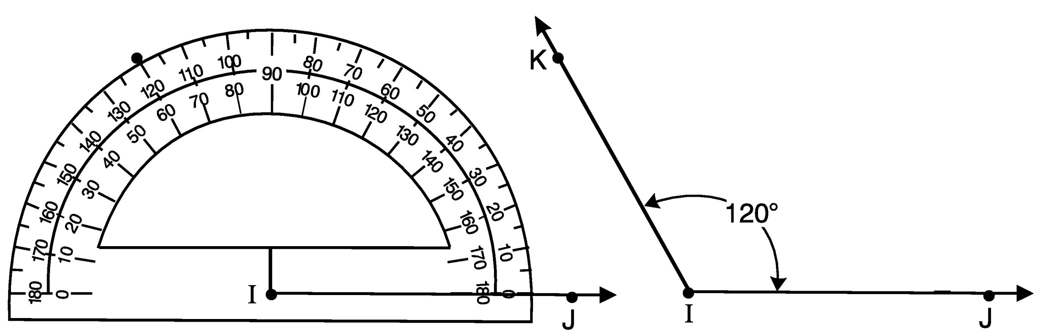
b. Align line $\overrightarrow{AB}$ with the 0° mark on the protractor. This example uses the outside scale.

c. The measure of ∠CAB is the number where line $\overrightarrow{AC}$ crosses the scale of the protractor. The measure of ∠CAB is equal to 60°. You can write this as ∠CAB = 60°.

Example 2

Draw an angle of 120°.

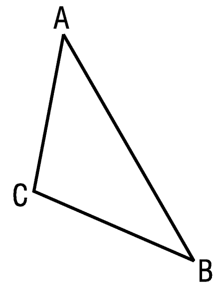
Solution



Draw a ray. Label the ray $\overrightarrow{IJ}$ Place the midpoint of the protractor over the endpoint I. Align $\overrightarrow{IJ}$ with the 0° mark on the outside scale. Make a point at 120°. Label the point K. Draw $\overrightarrow{IK}$.

Triangles

A triangle is a figure with three sides and three angles. The sum of the measures of the angles in a triangle is always equal to 180°.

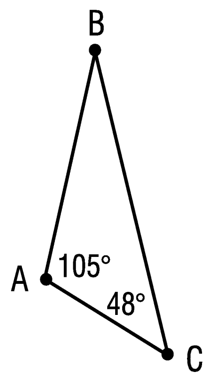


$$\angle A + \angle B + \angle C = 180°$$

Example 3

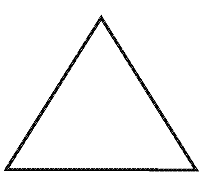
Find the measure of $\angle B$ in the figure below.

Solution

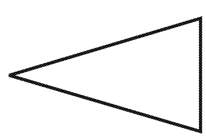


$$\angle A + \angle B + \angle C = 180°$$
$$105° + \angle B + 48° = 180°$$
$$\angle B + 153° = 180°$$
$$\angle B = 27°$$

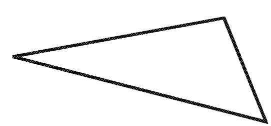
Triangles are often classified by the number of sides that are congruent.



Equilateral
3 congruent sides

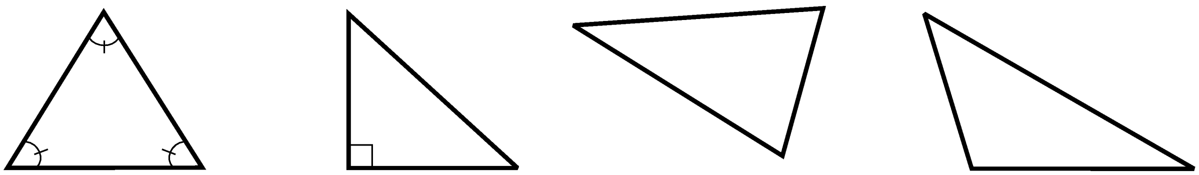


Isosceles
2 congruent sides



Scalene
No congruent sides

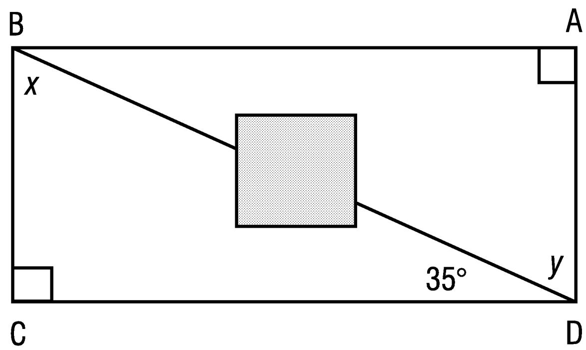
Triangles are also classified by their angles. Congruent angles are shown by the rounded marks on the angles below.



Equiangular 3 congruent angles | **Right** 1 angle measures 90°. | **Acute** Each angle measures less than 90°. | **Obtuse** 1 angle measure more than 90°.

Example 4

You are building a rectangular metal toolbox with a divider along the diagonal to keep tools sorted. The toolbox has a compartment in the middle for long narrow tools such as hammers, screwdrivers, and small saws. You determine that the angle of the divider at one corner of the box should be 35°. (a) What should be the measurement of $\angle x$ for the divider to be along the diagonal of the rectangle? (b) What then is the measure of angle $\angle y$? How does it relate to $\angle x$?



Solution

a. The sum of the angles of a triangle is 180°. A rectangle has four right angles. Therefore, the diagonal **BC** forms two right triangles, ΔBAD and ΔBCD. Use the given angle and the right angle to find $\angle x$.

$$\begin{aligned} 35° + \angle x + \angle C &= 180° \\ 35° + \angle x + 90° &= 180° \\ \angle x + 125° &= 180° \\ \angle x &= 55° \end{aligned}$$

The divider should make an angle of 55° with side **BC**.

b. As noted in part (a), a rectangle has four right angles. The originally given angle (35°) and $\angle y$ together make up one of these at $\angle D$. (We call them complementary angles.) Therefore we can make the following statement:

$$35° + \angle y = 90°$$

$$\angle y = 55°$$

Therefore, $\angle x = \angle y$.

Solution to Scenario Questions

Following are the solutions to the question posed under "Photonics Scenario Involving Geometry."

a. For tan B = tan $A_1 = n = 1.5$, so that tan $A_1 = n$;

$A_1 = \tan^{-1}(1.5)$ (Use the inverse tan key to evaluate.)

$A_1 \approx 56°$

b. $A_2 = A_1 = 56°$ (since "angle of reflection = angle of incidence")

c. Using Snell's law ($n_1 \sin A_1 = n_2 \sin B_1$) at first window surface;

$1 \times \sin A_1 = 1.5 \times \sin B_1$

$\sin B_1 = \dfrac{\sin A_1}{1.5}$

$\sin B_1 = \dfrac{\sin 56°}{1.5}$ (since from part (**a**) we found $A_1 \approx 56°$)

$\sin B_1 = 0.553$

$B_1 = \sin^{-1}(0.533)$ (Use the inverse sin key to evaluate.)

$B_1 = 34°$

d. From right triangle PQR

$\angle QRP = 90 - B_1$

$\angle QRP = 90 - 34$

$\angle QRP = 56°$

Since line SR is normal to the right surface,

$\angle QRP + B_2 = 90°$

$56° + B_2 = 90°$

$B_2 = 34°$

Angle B_3 is equal to angle B_2 from the law of reflection. So,

$B_3 = 34°$, $B_2 = 34°$ and

$B_3 = B_2 = B_1$

e. Using Snell's law ($n_2 \sin B_2 = n_4 \sin B_4$) at the second window surface,

$1.5 \sin B_2 = 1 \sin B_4$

$\sin B_4 = 1.5 \sin(34°)$

$\sin B_4 \approx 0.839$

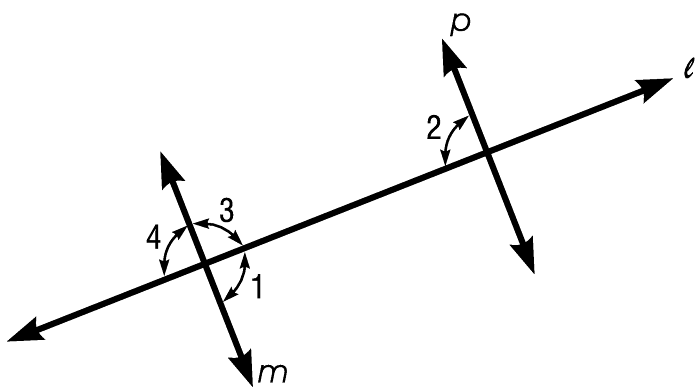
$B_4 \approx \sin^{-1}(0.839)$ (Use the inverse sin key to evaluate.)

$B_4 \approx 57°$

Practice Exercises

Exercise 1

Use the drawing below to answer the questions that follow:



a. If $\angle 1 = 90°$, what can you say about lines l and m?

b. If line p is perpendicular to line l ($p \perp l$), what is the measure of $\angle 2$?

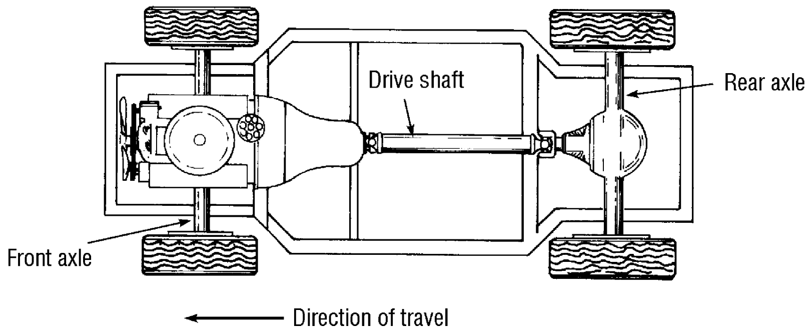
c. If $\angle 1 = 90°$, what is the measure of $\angle 3$?

d. If $\angle 1 = 90°$, what can you say about the measure of $\angle 4$?

e. Since $\angle 1$ and $\angle 2$ are right angles, what statement can you make about lines m and p?

Exercise 2

Below is a simple sketch of the frame and drive shaft for a rear-wheel drive automobile. Complete the statements that follow.



a. The drive shaft is ______ to the rear axle.

1. parallel
2. perpendicular
3. at an angle of 30°
4. at an angle of 45°

b. The rear axle and the front axle are ______ to each other.

1. parallel
2. perpendicular
3. at an angle of 30°
4. at an angle of 45°

c. The drive shaft is ______ to the direction of travel of the car.

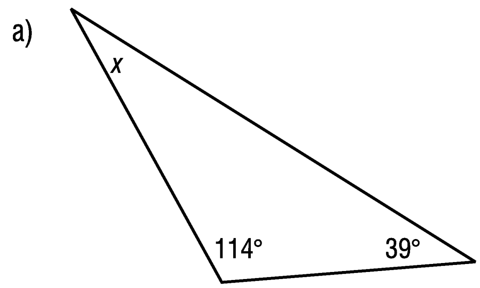
1. parallel
2. perpendicular
3. at an angle of 30°
4. at an angle of 45°

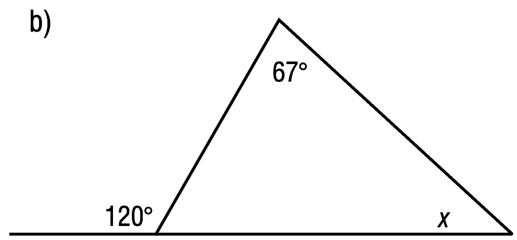
d. When the tires on the car have spun around 360° (one revolution), the distance they have moved on the ground is about the same as their ______.

1. radius
2. diameter
3. circumference
4. semicircle

Exercise 3

On the following triangles *solve* for $\angle x$.





Solutions to Practice Exercises

1. **a.** They are perpendicular $(l \perp m)$ because they intersect at a right angle (90°).

   b. $\angle 2 = 90°$; the reverse reasoning of part (a).

   c. $\angle 3 = 90°$; because $\angle 1$ and $\angle 3$ together make a straight angle (180°), so $\angle 1 + \angle 3 = 180° \Rightarrow 90° + \angle 3 = 180° \Rightarrow \angle 3 = 90°$.

   d. $\angle 4 = 90°$; if $\angle 1 = 90°$ then $\angle 3 = 90°$ (from part (c)), then since $\angle 3$ and $\angle 4$ make up a straight angle then $\angle 4 = 90°$.

   e. Lines m and p are parallel $(m \| p)$ because each line is perpendicular $(\perp)$ to the same line, l.

2. **a.** 2

   b. 1

   c. 1

   d. 3

3. **a.** $114° + 39° + x = 180°$

   $153° + x = 180°$

   $x = 27°$

   b. $(180° - 120°) + 67° + x = 180°$

   $60° + 67° + x = 180°$

   $127° + x = 180°$

   $x = 53°$

ANGLE MEASURES IN TWO AND THREE DIMENSIONS

Objectives

When you have completed this section, you should be able to do the following:

1. Convert back and forth from degrees to radians (two dimensions)
2. Calculate partial solid angles (three dimensions)

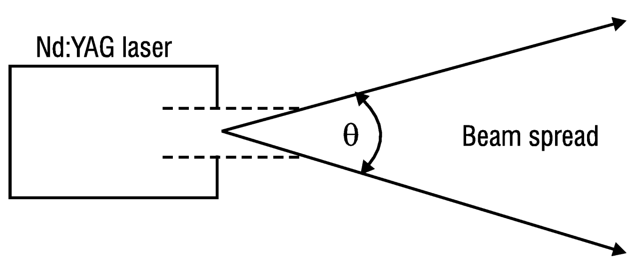
Photonics technicians need to perform calculations using angles measured in both degrees and radians, converting between the two, if necessary, by using the equivalency between 2π radians and 360 degrees. They also need to calculate partial solid angles (in three dimensions), such as the solid angle subtended by a beam of light emanating and spreading out from a point source of light.

Photonics Scenario Involving Angle Measures in Two and Three Dimensions

Laser light exiting an Nd:YAG laser passes through an exit aperture (opening) at the output laser mirror of diameter 1 millimeter. The wavelength of light for an Nd:YAG laser is 1.06 μm. You have learned that the beam spread of light of wavelength, λ, through a circular opening of diameter, D, is given by the equation:

$$\theta = 1.27\frac{\lambda}{D}$$

where θ is the beam angle spread (called divergence) and is given in radian measure.



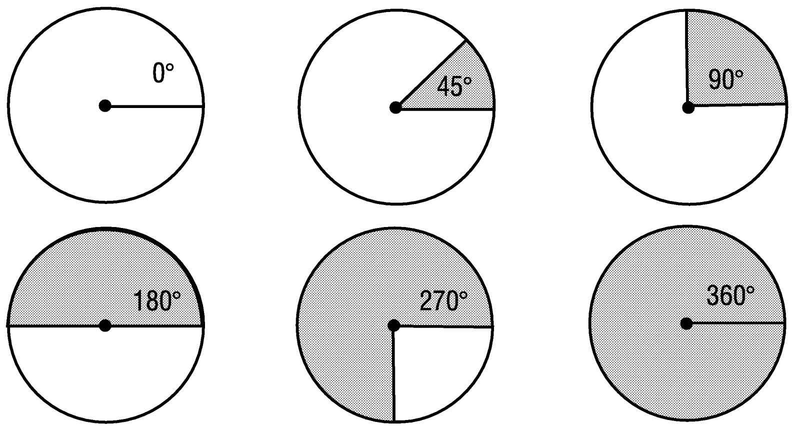
Question

For this laser, determine the beam spread angle in both radians and degrees.

Before looking at the solution, work through the lesson to further develop your skills in this area.

Radians and Degrees

In the previous section, we dealt extensively with angles and their measures in triangles. This section will explore the unique characteristics of angle measurement in two and three dimensions. Recall the following angle measures in circles.



These measures are in degrees. Another common unit of angle measure (particularly in circles) is *radians*. Just as there are 12 inches in a foot, there are 180° in π radians or 360° in 2π radians. This makes for a rather simple conversion as seen in the examples below.

Example 1

Convert 45° to radians.

$$\frac{45^\circ}{} \cdot \frac{\pi \text{ radians}}{180^\circ} = \frac{\pi}{4} \text{ radians}$$

Often *radians* is not used but is assumed. If "°" is used, it implies degrees. If there is no degree sign, radians should be assumed.

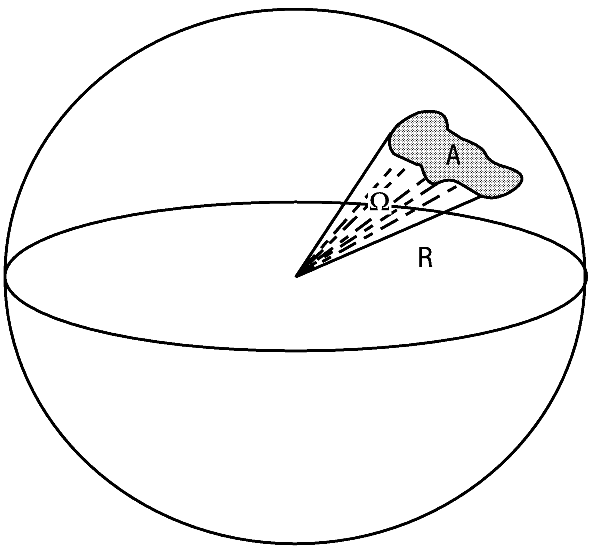
Example 2

Convert $\frac{3\pi}{2}$ to degrees.

$$\frac{3\cancel{\pi}}{2} \cdot \frac{180^\circ}{\cancel{\pi}} = 270^\circ$$

Solid Geometry

In solid geometry, the term "solid angle" is introduced by analogy with the idea of a plane angle in plane geometry. Thus, instead of two lines enclosing a plane angle, there now will be a conical surface enclosing a space. The enclosed space is called a *solid angle*. It is measured in *steradians* (abbreviated sr) and is usually denoted by the Greek letter omega (Ω). In plane geometry, 2π radians surround a point. In solid geometry, 4π steradians surround a point.



$$\Omega = \frac{\text{area intercepted}}{(\text{radius})^2}$$

$$\Omega = \frac{A}{R^2} \text{ in steradians}$$

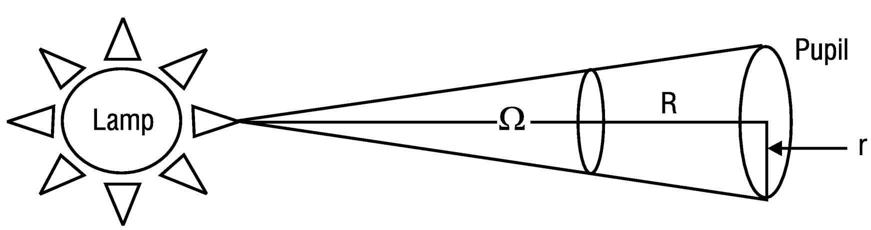
$$\Omega \text{ sphere} = \frac{4\pi R^2}{R^2} \quad \leftarrow \text{(surface area of a sphere)}$$

$$\Omega \text{ sphere} = 4\pi \text{ steradians}$$

The above figure shows a solid angle, Ω, defined in terms of the intercepted area on the spherical surface, A, and the radius of the sphere, R.

Example 3

Calculate the solid angle, Ω, that would be intercepted by the pupil of a man's eye (7-mm radius) from a lamp 2 meters away.



Solution

$$2\ \cancel{\text{m}} \times \frac{1000\ \text{mm}}{1\ \cancel{\text{m}}} = 2000\ \text{mm}$$

$$\Omega = \frac{\pi r^2}{R^2} = \frac{(\text{area of pupil})}{(\text{distance from lamp to pupil})^2}$$

$$\Omega = \frac{\pi(7\ \text{mm})^2}{(2000\ \text{mm})^2}$$

$$\Omega = 3.85 \times 10^{-5}\ \text{steradians}$$

Solution to Scenario Question

Following is the solution to the question posed under "Photonics Scenario Involving Angle Measures in Two and Three Dimensions."

Beam spread angle θ in radians is given by $\theta = 1.27\frac{\lambda}{D}$.

$$\text{Therefore, } \theta_{\text{rad}} = \frac{(1.27)(1.06 \times 10^{-6}\ \cancel{\text{m}})}{1 \times 10^{-3}\ \cancel{\text{m}}} = 1.35 \times 10^{-3}\ \text{radians}$$

$$= 1.35\ \text{milliradians}$$

Now, converting to degrees:

$$\frac{1.35 \times 10^{-3}\ \cancel{\text{rad}}}{} \left| \frac{180°}{\pi\ \cancel{\text{rad}}} \right. = 0.077°$$

The beam spread of laser beams (here less than one tenth of a degree!) is much smaller than that of flashlights, searchlights, and so on.

Practice Exercises

Exercise 1

Answer the following questions about angles and measuring angles.

a. How many degrees are in a circle?

b. If a pie is cut in half (across a diameter), then each of those halves is cut in half, and then each of those pieces is cut in half, how many pieces of pie are cut? How many degrees does each piece span?

c. How many degrees are between each of the numbers on the face of a clock? (That is, how many degrees are between the 12 and the 1, between the 1 and the 2, and so on?)

d. On the face of a clock, if you start at 12 o'clock and move clockwise, how many degrees will you cross between 12 and 3? Between 12 and 6? Between 12 and 9? Between 12 and all the way around to 12 again?

Exercise 2

Convert 60° to radians.

Exercise 3

Convert 150° to radians.

Exercise 4

Convert $\frac{\pi}{3}$ to degrees.

Exercise 5

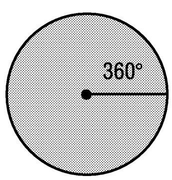
Convert $\frac{5\pi}{4}$ to degrees.

Exercise 6

Calculate the solid angle, Ω, that would be intercepted by a rectangular mirror (4 cm by 6 cm) from a lamp 1.5 meters away.

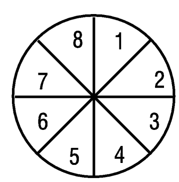
Solutions to Practice Exercises

1. **a.** 360°



b. 8 sections (equal size)

$$\frac{360°}{8} = 45° \text{ per piece}$$



c. There are 12 sections to a clock. So,

$$\frac{360°}{12} = 30°$$

d. 12 and 3

3 sections $3 \times 30° = 90°$ (forms a right angle)

12 and 6

6 sections $6 \times 30° = 180°$ (forms a straight angle)

12 and 9

9 sections $\quad 9 \times 30° = 270°$

12 to 12

12 sections $\quad 12 \times 30° = 360°$ (forms a full circle)

2.

$$\begin{array}{c|c} 60° & \pi \\ \hline & 180° \end{array} = \frac{\pi}{3} \text{ radians}$$

3.

$$\begin{array}{c|c} 150° & \pi \\ \hline & 180° \end{array} = \frac{5\pi}{6} \text{ radians}$$

4.

$$\begin{array}{c|c} \cancel{\pi} & 180° \\ \hline 3 & \cancel{\pi} \end{array} = 60°$$

5.

$$\begin{array}{c|c} 5\cancel{\pi} & 180° \\ \hline 4 & \cancel{\pi} \end{array} = 225°$$

6.

$$\begin{array}{c|c} 1.5 \ \cancel{\text{m}} & 100 \text{ cm} \\ \hline & 1 \ \cancel{\text{m}} \end{array} = 150 \text{ cm}$$

$$\Omega = \frac{l \times w}{R^2}$$

$$\Omega = \frac{4 \text{ cm} \times 6 \text{ cm}}{(150 \text{ cm})^2}$$

$$\Omega = \frac{24 \ \cancel{\text{cm}^2}}{22500 \ \cancel{\text{cm}^2}}$$

$$\Omega = 1.07 \times 10^{-3} \text{ steradians}$$

TRIGONOMETRY

Objectives

When you have completed this section, you should be able to do the following:

1. Apply the Pythagorean theorem to solve problems
2. Use trigonometry functions (and their inverses) and know when to apply them

Photonics technicians frequently need to recognize problems that can be solved by using the Pythagorean formula. In addition, they use the trigonometric functions of sine, cosine, and tangent with a scientific calculator or computer to solve geometric problems involving angles and distances such as found in Snell's Law of Refraction, $n_1 \sin\theta_1 = n_2 \sin\theta_2$.

Photonics Scenario Involving Trigonometry

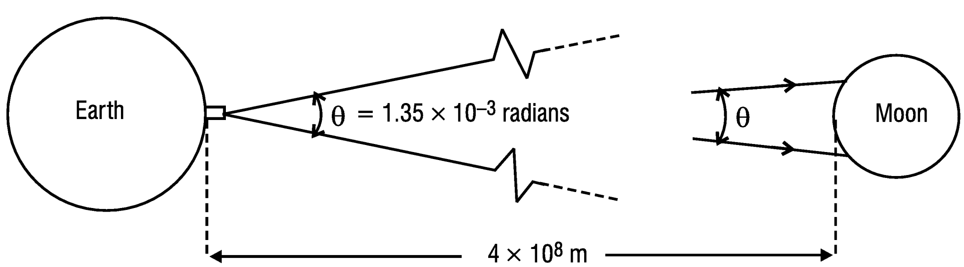
U.S. astronauts placed a retroreflector panel of dimensions one meter by one meter on the surface of the moon. On the earth, a team of scientists pointed an Nd:YAG laser of beam divergence $\theta = 1.35$ milliradians at the panel. Take the distance from the laser on the earth to the panel on the moon to be roughly 4×10^8 meters.

Questions

a. What is the diameter of the circular spot of the Nd:YAG laser beam that strikes the panel on the moon?

b. If the Nd:YAG laser emits a beam of 5×10^7 watts (a 50-joule pulse in 1 microsecond), what is the power per unit area reaching the retroreflector?

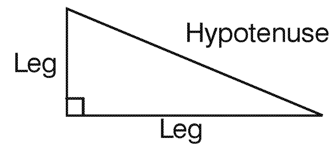
c. What is the solid angle Ω that the 1-m^2 panel subtends back at the laser on the earth?



Before looking at the solutions, work through the lesson to further develop your skills in this area.

Key Trigonometry Concepts

A **right triangle** is a triangle that contains a right angle. The **legs** of the triangle form the right angle. The side opposite the right angle is the **hypotenuse**. The legs and the hypotenuse are related by the Pythagorean theorem.

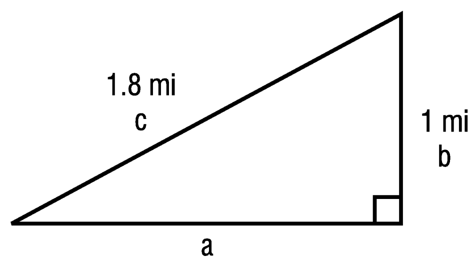


$$(\mathbf{Leg_1})^2 + (\mathbf{Leg_2})^2 = (\mathbf{Hypotenuse})^2$$

Example 1

You reset your trip odometer at the bottom of a hill and then proceed to drive up the hill until you come to a sign that says you are 1 mile high. Your trip odometer reading says you have driven 1.8 miles. How far horizontally are you from your initial starting point?

It often helps to draw a picture to represent the problem. Here the hypotenuse, c, represents the ground.

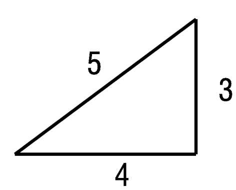


Use the Pythagorean theorem.

$$\begin{aligned} a^2 + b^2 &= c^2 \\ a^2 + (1.0\text{ mi})^2 &= (1.8\text{ mi})^2 \\ a^2 + 1.0\text{ mi}^2 &= 3.24\text{ mi}^2 \\ a^2 &= 2.24\text{ mi}^2 \\ a &= \sqrt{2.24\text{ mi}^2} = 1.5\text{ mi (rounded)} \end{aligned}$$

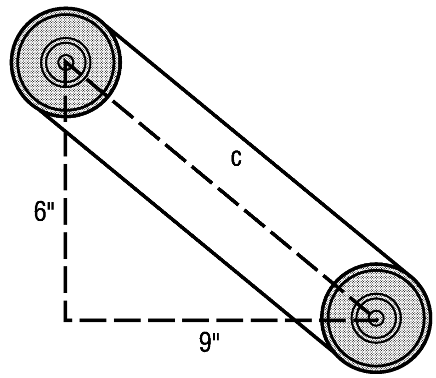
You are 1.5 miles horizontally from your starting point.

The whole numbers a, b, and c that satisfy $a^2 + b^2 = c^2$ are called a Pythagorean triple. For example, (3, 4, 5) is a Pythagorean triple because $3^2 + 4^2 = 5^2$. Triangles with sides having lengths equal to Pythagorean triples are useful for solving problems.



Example 2

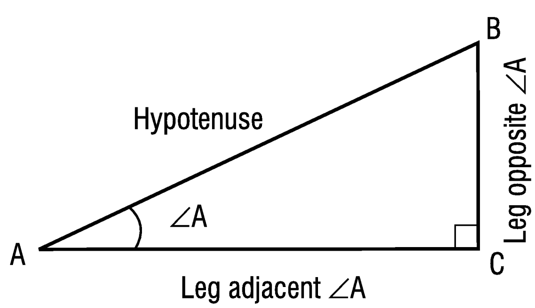
What is the distance between the centers of two pulleys if one is placed 9.0" to the left and 6.0" above the other.



Solution

$$a^2 + b^2 = c^2$$
$$(6.0\text{ in})^2 + (9.0\text{ in})^2 = c^2$$
$$36\text{ in}^2 + 81\text{ in}^2 = c^2$$
$$117\text{ in}^2 = c^2$$
$$\sqrt{117\text{ in}^2} = c^2$$
$$11\text{ in (rounded)} = c^2$$

The measurements of the angles and sides of right triangles are related through the ratios of **trigonometry**. As shown in the illustration here, an acute angle of a right triangle has an adjacent leg and an opposite leg.



The tangent, sine, and cosine ratios for this angle are defined as follows:

$$\text{Tangent of } \angle A = \frac{\text{Length of leg opposite } \angle A}{\text{Length of leg adjacent to } \angle A}$$

$$\tan A = \frac{\text{opp}}{\text{adj}}$$

$$\text{Sine of } \angle A = \frac{\text{Length of leg opposite } \angle A}{\text{Length of hypotenuse}}$$

$$\sin A = \frac{\text{opp}}{\text{hyp}}$$

$$\text{Cosine of } \angle A = \frac{\text{Length of leg adjacent to } \angle A}{\text{Length of hypotenuse}}$$

$$\cos A = \frac{\text{adj}}{\text{hyp}}$$

Example 3

You need to determine the angle of a car's rear window with the vertical so you can cut a brake-light shield to match the angle. You measure the depth of the rear ledge as 12 inches and its corresponding height as 10½ inches.

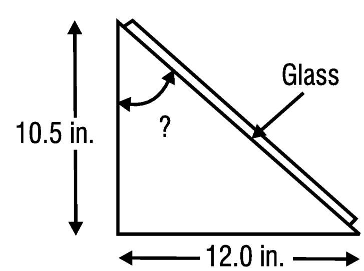
a. Make a sketch of the right triangle formed by the rear window. Label the known dimensions. Identify the angle that the window surface makes with the vertical.

b. Compute the value of the tangent of this angle by using the ratio of the lengths of the opposite and adjacent legs.

c. When the value of the tangent of an angle is known, the *inverse tangent* function (on your calculator) will give the value of the angle. Simply enter the value of the tangent (that is, the ratio) into your calculator and press the $\tan^{-1}$ key OR the *INV* key followed by the *TAN* key—the angle will be in the display. Do this to find the value of the angle for your rear window.

Solution

a. The sketch should appear generally as shown below.



b. The tangent of the angle is the ratio of the length of the opposite leg to that of the adjacent leg. It is important that you correctly identify the angle and the corresponding "opposite" and "adjacent" legs.

$$\text{Tangent of angle} = \frac{\text{Opposite leg}}{\text{Adjacent leg}}$$

$$\text{Tangent of angle} = \frac{12.0 \text{ in}}{10.5 \text{ in}}$$

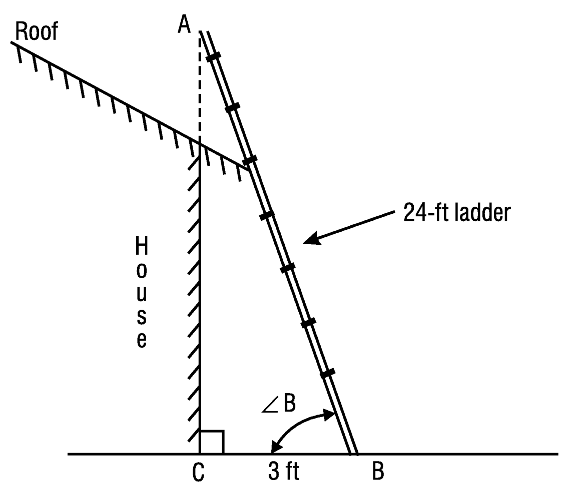
$$\text{Tangent of angle} = 1.14286\ldots$$

c. Glass angle = *INV TAN* (tangent of angle)
Glass angle = *INV TAN* (1.14286)
Glass angle = 48.8° (rounded)

Note: Some calculators must be set to "degree mode" to yield angles in degrees, rather than radians, for example. Also, the inverse tangent function may be labeled "$\tan^{-1}$."

Example 4

For safety purposes, a ladder manufacturer recommends that the angle a ladder makes with the ground should be greater than 65° but less than 80°. To repair a leak near the edge of the roof on your house, you need to extend a 24-ft ladder to its full length. When you do this, the base of the ladder is 3 ft from the vertical wall, as shown in the illustration. Can you safely use this ladder to make the repair? What are the minimum and maximum base distances for safe ladder use?



Solution

You need to find whether $65° < \angle B < 80°$. For the right triangle ACB, you know the lengths of the hypotenuse and the side adjacent to $\angle B$. Therefore, use the cosine ratio.

$$\cos B = \frac{\text{adj}}{\text{hyp}}$$

$$\cos B = \frac{3}{24}$$

$$\cos B = 0.125$$

To find $\angle B$, use the inverse cosine ($\cos^{-1}$). You can use your calculator.

$$0.125 \; \boxed{\text{2nd}} \; \boxed{\cos^{-1}} = \boxed{82.81924422}$$

Since $\angle B \geq 80°$, you **cannot** safely use the ladder in this position.

To find the minimum and maximum base distances, again use the cosine relationship. Solve for the adjacent length using the minimum and maximum values of angle B.

$$\cos B = \frac{\text{adj}}{\text{hyp}}$$

$$\cos 65° = \frac{\text{adj}}{24 \text{ ft}}$$

$$\text{adj} = 24 \text{ ft} \bullet \cos 65°$$

$$\text{adj} = 10.14 \text{ ft, or } 10 \text{ ft (rounded)}$$

$$\cos B = \frac{\text{adj}}{\text{hyp}}$$

$$\cos 80° = \frac{\text{adj}}{24 \text{ ft}}$$

$$\text{adj} = 24 \text{ ft} \bullet \cos 80°$$

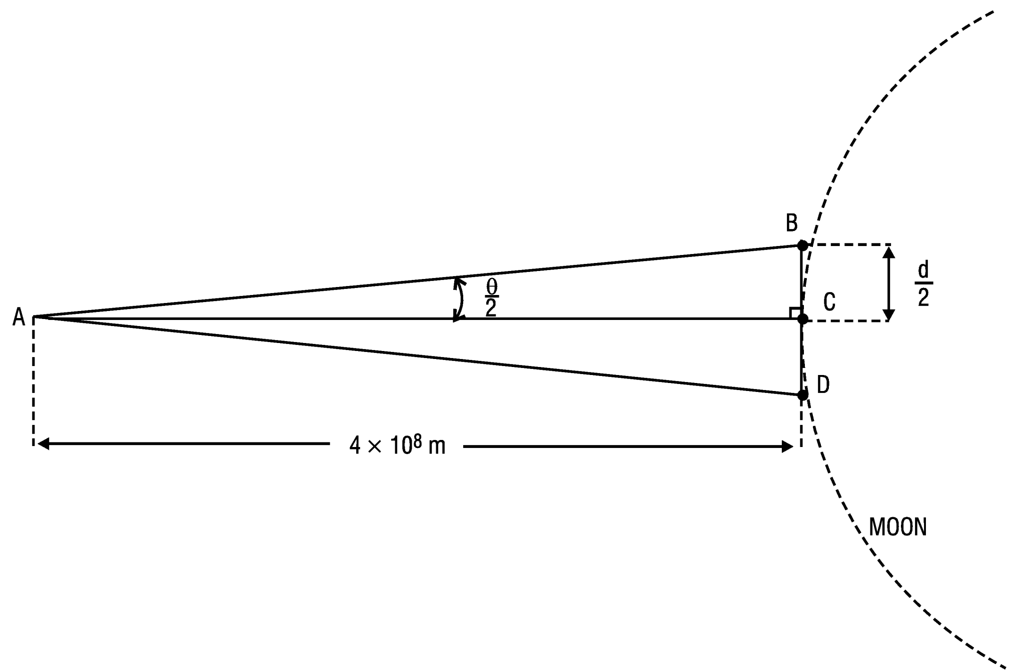
$$\text{adj} = 4.168 \text{ ft, or } 4.2 \text{ ft (rounded)}$$

The base of the fully extended ladder should be no closer to the wall than 4.2 ft and no farther from the wall than 10 ft.

Solution to Scenario Question

Following are the solutions to the questions posed under "Photonics Scenario Involving Trigonometry."

a.



Point **A** is the origin of the laser on the earth.
Segment **BD** is the diameter of the laser spot when it strikes the moon.

From the right triangle ACB,

$$\tan\frac{\theta}{2} = \frac{BC}{AC}$$

$$\tan\frac{\theta}{2} = \frac{\frac{d}{2}}{4 \times 10^8 \text{ m}}$$

$$\therefore \frac{d}{2} = (4 \times 10^8 \text{ m})(\tan\frac{1.35 \times 10^{-3}\text{rad}}{2})$$

(With the calculator set to radians, use the tan key to evaluate.)

$$\frac{d}{2} = (4 \times 10^8 \text{ m})(6.75 \times 10^{-4} \text{ rad})$$

$$\frac{d}{2} = 2.7 \times 10^5 \text{ m}$$

$$d = 5.4 \times 10^5 \text{ m} = 540 \text{ km}$$ (MUCH larger than the 1-m^2 retroreflector)

b. Power per unit area $= \dfrac{\text{Incident power}}{\text{Area of spot}}$

$$= \frac{5 \times 10^7 \text{ W}}{\pi(\frac{d}{2})^2 \text{ m}^2}$$

$$= \frac{5 \times 10^7 \text{ W}}{\pi(2.7 \times 10^5)^2 \text{ m}^2}$$

Power per unit area $= 2.2 \times 10^{-4}$ W/m^2

c. $\Omega = \dfrac{A_\perp}{R^2} = \dfrac{1 \text{ m}^2}{(4 \times 10^8 \text{ m})^2} = 6.25 \times 10^{-18}$ steradians (very small!)

Practice Exercises

Exercise 1

In electrical circuits with varying currents and voltages, the combined effect of resistance and reactance is called impedance. The impedance is related to the resistance and the reactance by the formula shown below.

$$Z^2 = R^2 + X^2$$

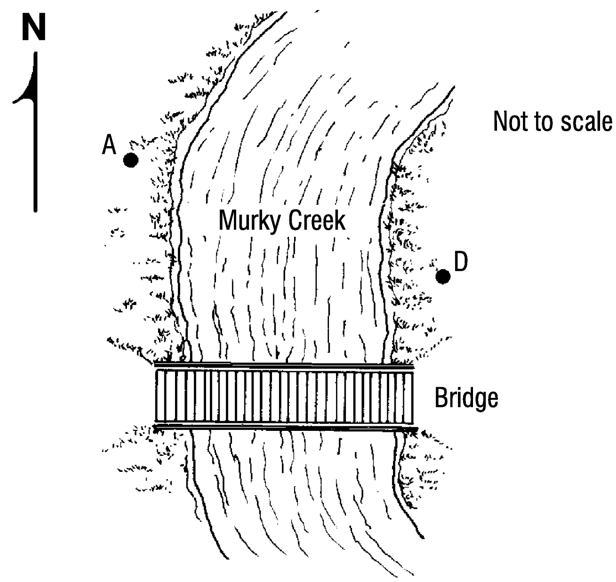
where Z is the impedance, in ohms,
R is the resistance, in ohms, and
X is the reactance, also in ohms.

a. The relationship above is equivalent to the Pythagorean formula that relates the sides of a right triangle. Draw and label a right triangle to represent the relation among the impedance, the resistance, and the reactance.

b. A loudspeaker is labeled as having an impedance of 8.0 ohms. Your measurement with an ohmmeter shows a resistance of 1.5 ohms. What is the reactance of the speaker?

Exercise 2

You need to string a cable across Murky Creek and need to determine the distance across. Suppose the two points can be labeled A and D, as shown in the sketch below. You find that, starting at point A, you can walk due south for 21 meters then turn due east and walk 22.5 meters across the bridge. Finally, walking due north 9 meters you reach the desired point D.

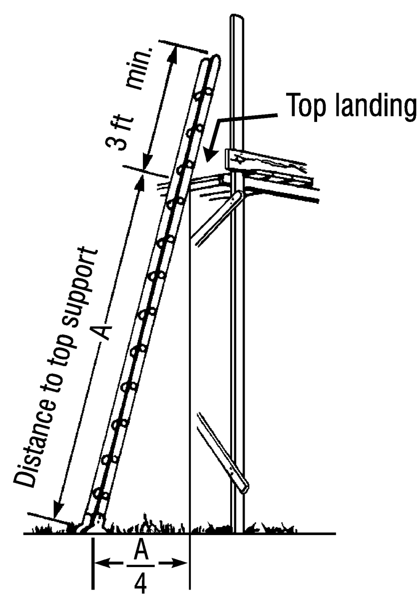


a. On your paper, make a sketch of the distances and directions traveled, similar to the sketch above. The final point should be labeled D.

b. The segment joining point A to point D on your scale drawing is the distance you need to string cable. How many meters is it across Murky Creek, from point A to point D?

Exercise 3

A manual has the illustration shown here as a guideline for the safe use of an extension ladder. Suppose your ladder has a maximum extended length of 24 feet.



a. Using the guideline of having 3 feet extending above the top landing, what is the distance to the top support for your ladder?

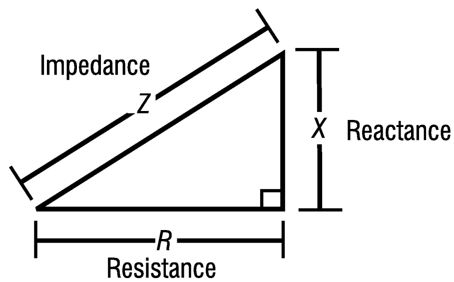
b. Using the guideline shown in the drawing, what is the maximum landing height that you can safely reach with your ladder?

Exercise 4

You want to build a picture frame from a kit. Each of the four frame pieces is 12 inches long (on the longest edge). To check for "squareness" you plan to measure the diagonals. How long should the diagonal measurement be (at its longest point) for this frame?

Solutions to Practice Exercises

1. a. The labels for the legs can be reversed, but the hypotenuse must correspond to the impedance Z.

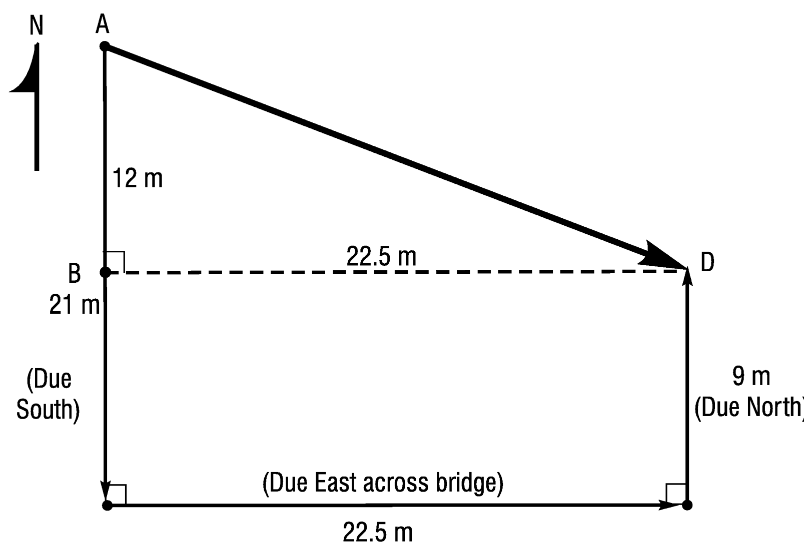


b. $X^2 = Z^2 - R^2$

$$X = \sqrt{(8.0 \text{ ohms})^2 - (1.5 \text{ ohms})^2}$$

$$X = 7.9 \text{ ohms (rounded)}$$

2. a.



b. Using the right triangle ABD shown in the drawing above:

$$AD = \sqrt{(AB)^2 + (BD)^2}$$

$$AD = \sqrt{(21 - 9)^2 + (22.5)^2}$$

$$AD = \sqrt{12^2 + (22.5)^2}$$

$$AD = 25.5 \text{ m}$$

3. **a.** To have 3 feet above the top landing with a 24-foot ladder, you can have no more than 21 feet (that is, 24 ft – 3 ft) between the point at which the ladder touches the ground and the top support.

b. The drawing with the exercise shows a right triangle whose base is one-quarter the length of the hypotenuse. The hypotenuse was determined in part (a) to be 21 feet. Then use the Pythagorean formula to solve for the remaining leg.

$c^2 = a^2 + b^2$(Solve for one leg, b.)

$$b = \sqrt{c^2 - a^2}$$

Here $c = 21$ ft and $a = \frac{c}{4} = \frac{21}{4} = 5.25$ ft

$$b = \sqrt{(21)^2 - (5.25)^2}$$

$b = 20.33$ ft or 20' 4" (rounded)

This ladder's maximum safe landing height is 20' 4" from the ground.

4. Since the frame will be square, each pair of sides forms a right triangle. The diagonal will be the hypotenuse, so you can use the Pythagorean formula.

$$c^2 = a^2 + b^2$$

$$c = \sqrt{(12 \text{ in})^2 + (12 \text{ in})^2}$$

$c = \sqrt{288 \text{ in}^2}$, or 17 in (rounded)

Or you may use the fact that the right triangle formed is a 45°– 45° triangle. Then the hypotenuse is $\sqrt{2}$ times the length of one leg, or $12\sqrt{2}$ inches, yielding the same answer as above.

SPECIAL GRAPHS

Objectives

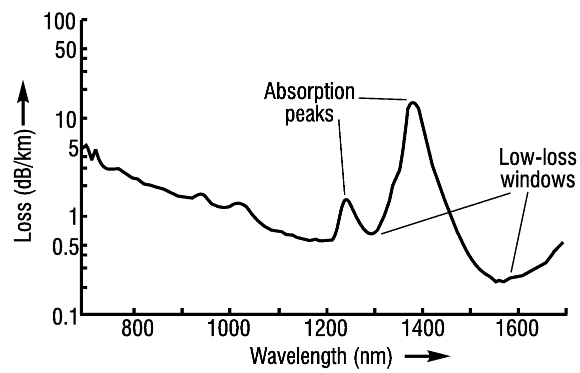
When you have completed this section, you should be able to do the following:

1. Read and interpret exponential graphs on linear and semilog scales
2. Graph exponential functions on semilog scales
3. Read and interpret polar graphs
4. Graph with polar coordinates
5. Convert back and forth from rectangular to polar coordinates

Photonics technicians need to draw, read, and interpret graphs of wide-ranging data and exponential relationships on either linear or semilog scales. Similarly, they need to draw, read, and interpret polar graphs, and use trigonometric properties to convert these to rectangular coordinates. For example, when portraying the fall-off in intensity from a light source, technicians generally measure and plot the light intensity at a distance r and angle θ from a reference point and line on a polar graph of r and θ.

Photonics Scenario Involving Special Graphs

The figure to the right shows how the power loss per kilometer (dB/km) of a typical silica optical fiber varies with the wavelength of light transmitted by the fiber. Note that the graph is a *semilog plot.*



Questions

a. What is the power loss per kilometer of optical fiber around 800 nm? Around 1550 nm?

b. At what wavelength is the power loss per kilometer the largest? What is the value of this loss in dB/km?

Before looking at the solutions, work through the lesson to further develop your skills in this area.

Graphs of Exponentials

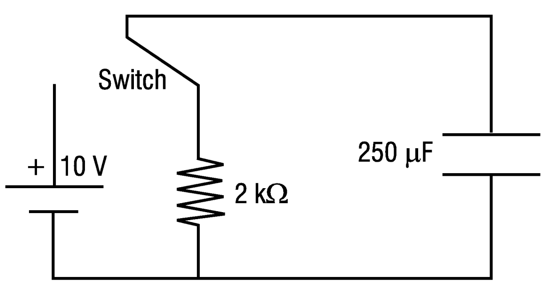
We have discussed exponential functions in previous sections, but now we will look at their graphical representations. The graphs of exponential functions differ from linear functions in that they do not have constant increases or decreases of y as x changes. In other words, there is not a constant slope, as is true with linear graphs. We will begin our study of exponential functions with a look at the graph of a common relationship in electrical circuit analysis involving the discharging of a capacitor.

$$V = V_0 e^{-t/RC} \text{ or } \frac{V}{V_0} = e^{-t/RC}$$

where V = Voltage across capacitor C after time (t)
V_0 = Battery voltage
t = Time since charging process began
R = Resistance in the series circuit
C = Capacitance in the series circuit
V/V_0 = Voltage ratio

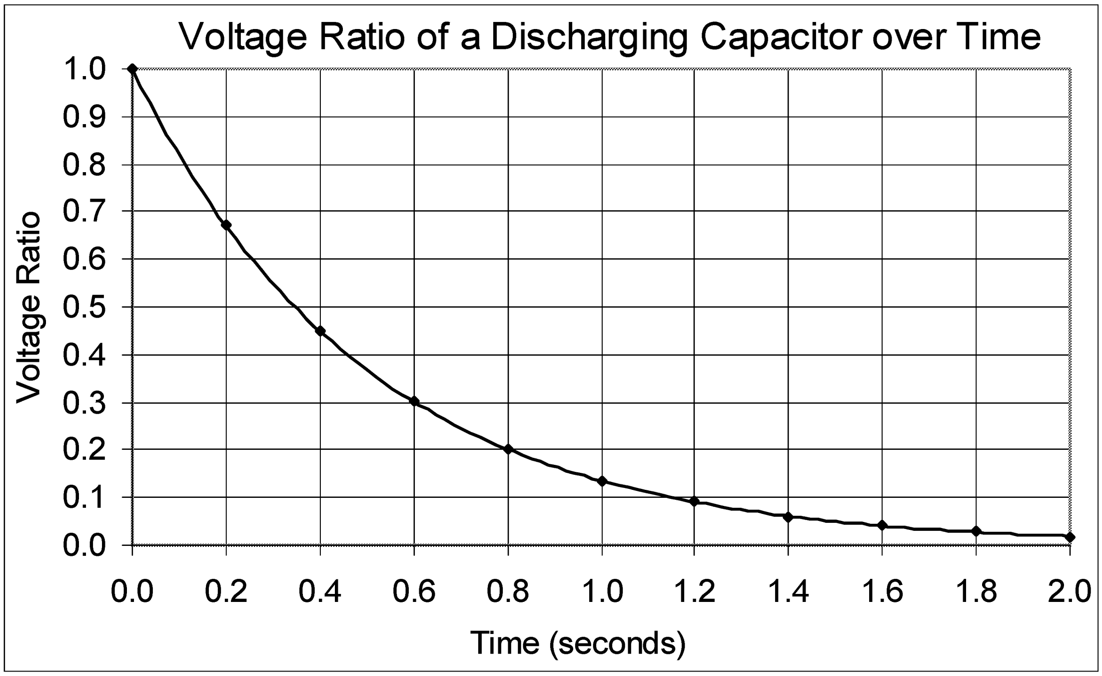
Example 1

A 10-volt (V_0 = 10 V) battery was used to charge a 250-µF capacitor. It is going to be discharged through a 2-kΩ resistor. Below are a circuit diagram and a list of measured values of the voltage ratio (V/V_0) as the capacitor **discharges**.



Time (seconds)	Voltage Ratio
0.0	1
0.2	0.670
0.4	0.449
0.6	0.301
0.8	0.202
1.0	0.135
1.2	0.091
1.4	0.061
1.6	0.041
1.8	0.027
2.0	0.018

Now we can plot these points in the Cartesian coordinate system with time on the x-axis and voltage on the y-axis.



Initially the voltage ratio rapidly decreases, but as it approaches zero the decrease slows. We say that it approaches zero *asymptotically*. The rate at which it decreases is called the *time constant* (τ). It can be calculated by multiplying the resistance by the capacitance ($R \times C$). Now we can rewrite the equation stated earlier:

$$\frac{V}{V_0} = e^{-t/\tau}$$

This simplified version deals only with our variables (time and voltage ratio) and the time constant (τ). You will learn more about the properties of the time constant and the uses of this equation in another course. For now you can associate the time constant with exponential functions as you do the slope with linear functions.

Example 2

The intensity level of sound is modeled by a logarithmic function.

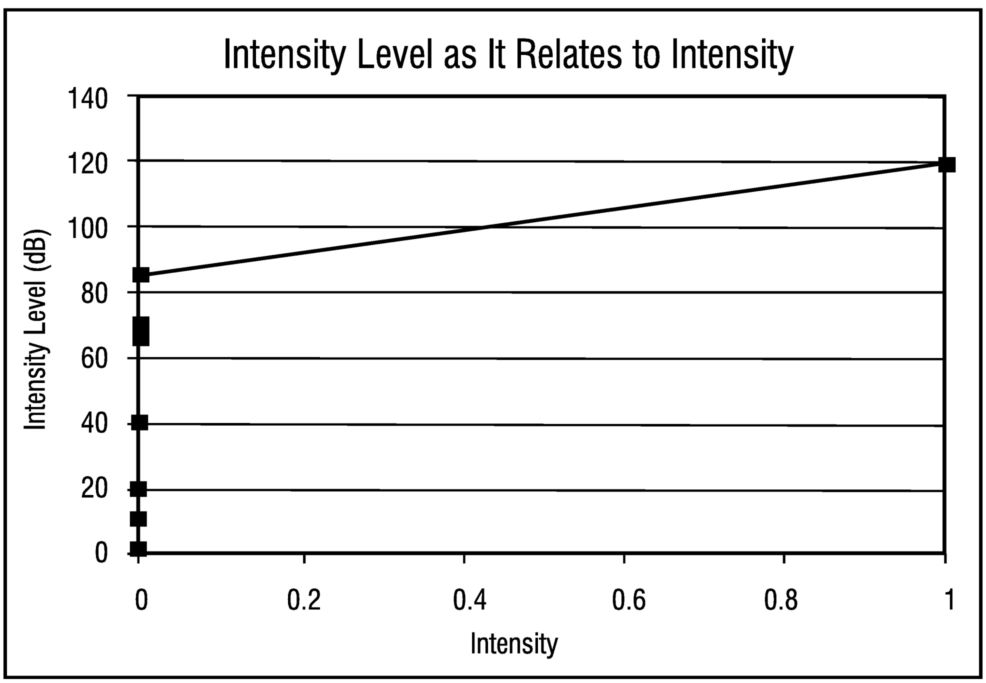
$$IL = 10\log_{10}\left(\frac{I}{I_0}\right)$$

where IL = Intensity level (dB)
I = Intensity (W/m^2) of a particular sound
I_0 = Standard threshold of audibility (hearing) = 1×10^{-12} (W/m^2)

The standard threshold of audibility (I_0) is a constant value measured at 10^{-12} W/m^2. For given intensities, the following intensity levels were measured as seen in the table below:

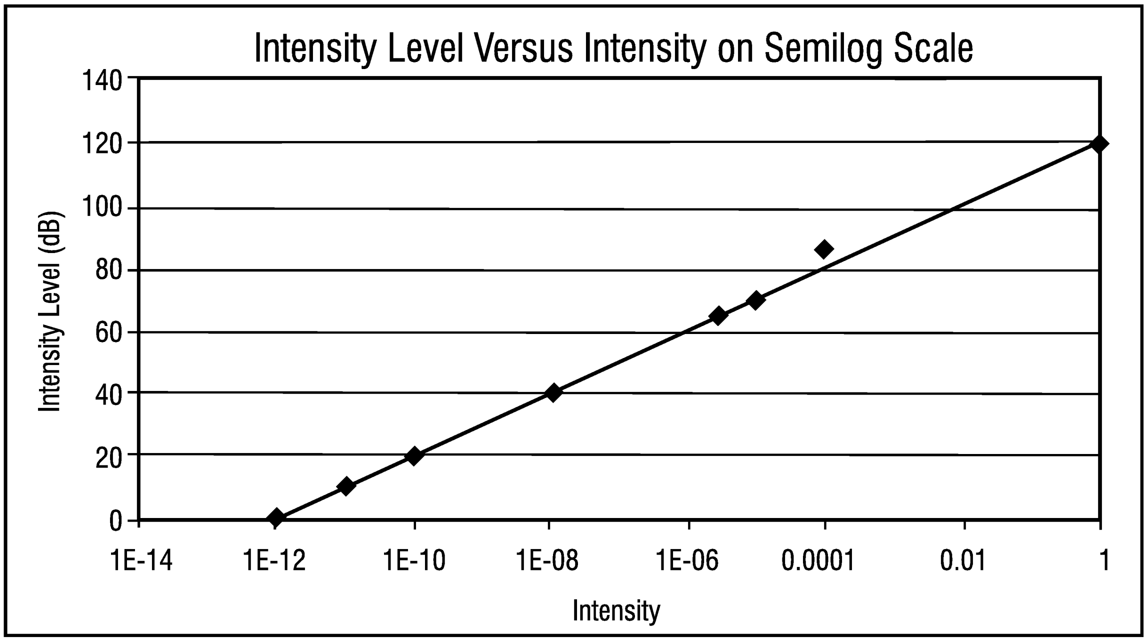
Sound	Intensity (W/m^2)	Intensity Level (dB)
Painful	1	120
Train	1×10^{-4} (or 1E-04)	85
Busy street traffic	1×10^{-5} (or 1E-05)	70
Typical conversation	3×10^{-6} (or 3E-06)	65
Quiet radio	1×10^{-8} (or 1E-08)	40
Whisper	1×10^{-10} (or 1E-10)	20
Rustle of leaves	1×10^{-11} (or 1E-11)	10
Hearing threshold	1×10^{-12} (or 1E-12)	0

From this table, we can attempt a graph with linear scales along the *x*- and *y*-axes. But, as the plot shows below, this does not come without difficulties.



The problem with this graph is that many of the *x*-values are so small that they all appear to be zero. If you were to drop the point (1, 120) you would have the same difficulty with the largest point being visible and all the others apparently on the *y*-axis. Although this graph *is* a function, it does not appear to be one. So in this case the graph is of little use.

There is an alternative though. We can represent the data on a semilog plot. This means one axis (x **or** y) is "logged." In this problem the x-axis is on a logarithmic scale as seen below:



This new plot results in a straight line! What is most useful about this graph is that the points are distinguishable. However, it may be difficult to see how the "logged" axis is working. Below are a table of arbitrary (yet simple) data and the accompanying graphs on a regular (both axes linear) scale and a semilog (one axis linear and one logarithmic) scale.

Example 3

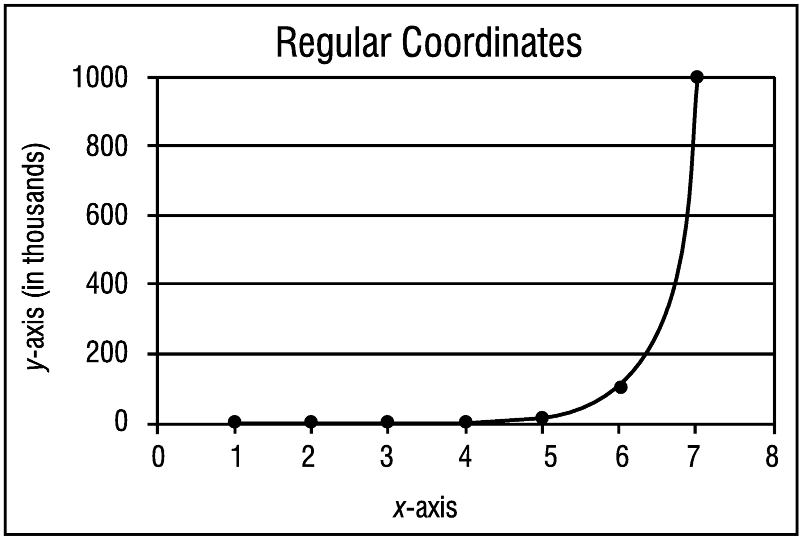
Plot the following table of data with:

a. 2 linear scales.

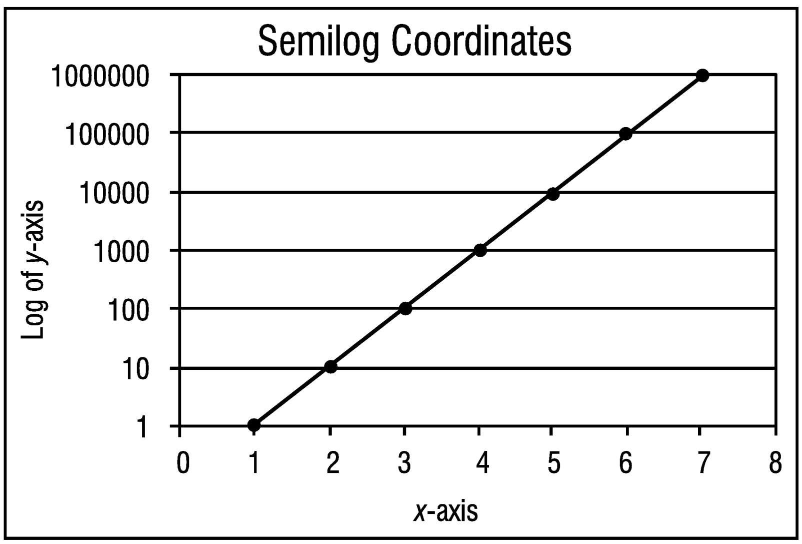
b. The x-axis as linear and the y-axis as logarithmic (semilog).

x	y
1	1
2	10
3	100
4	1000
5	10000
6	100000
7	1000000

a.



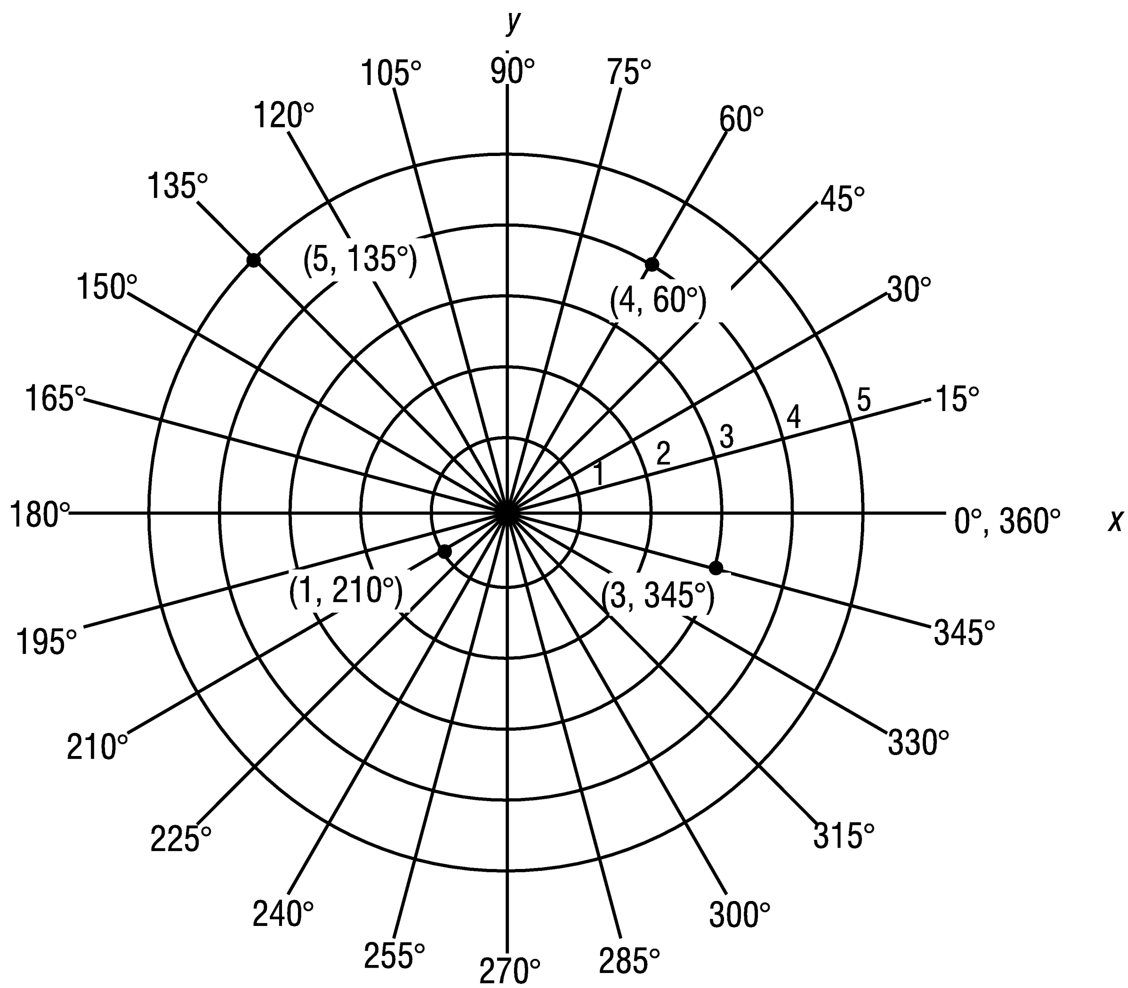
b.



This is another situation in which the semilog plot is better because the information is much easier to read.

Polar Coordinates

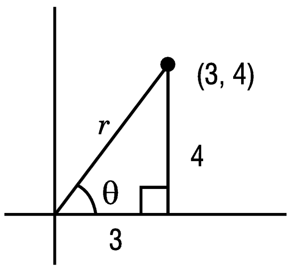
Another important coordinate system is polar coordinates. It has many important uses. The basis for this system is that points are plotted, not in (x, y), but in (r, θ). The first coordinate, r, is the radius and measures the distance from the origin to the point. The second, θ, is the angle measured counterclockwise from the positive x-axis as practiced in the angle measurement section of this book. The figure below shows some points plotted in polar coordinates.



Conversion

Rectangular to Polar

Oftentimes, we need to convert from rectangular coordinates to polar and vice versa. The process of converting the Cartesian point (3, 4) to polar coordinates is demonstrated below.



$$r^2 = 3^2 + 4^2 \qquad \text{(Pythagorean theorem)}$$
$$r = 5$$

$$\tan\theta = \frac{4}{3} \qquad \left(\tan\theta = \frac{\text{opposite}}{\text{adjacent}}\right)$$
$$\theta = \tan^{-1}\left(\frac{4}{3}\right)$$
$$\theta = 53.1°$$

Therefore, (3, 4) in rectangular coordinates is equivalent to (5, 53.1°) in polar coordinates.

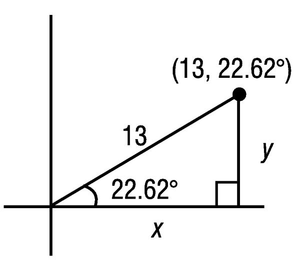
$$(3, 4) \Leftrightarrow (5, 53.1°)$$

In general, we can now make the following assumptions:

$$r = \sqrt{x^2 + y^2} \quad \text{AND} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Polar to Rectangular

Converting from polar to rectangular coordinates is another important skill. Here we will convert the polar point (13, 22.62°) to Cartesian coordinates.



$$\cos 22.62° = \frac{x}{13} \qquad \left(\cos\theta = \frac{\text{adjacent}}{\text{hypotenuse}}\right)$$
$$x = 13\cos 22.62°$$
$$x = 12$$

$$\sin 22.62° = \frac{y}{13} \qquad \left(\sin\theta = \frac{\text{opposite}}{\text{hypotenuse}}\right)$$
$$y = 13\sin 22.62°$$
$$y = 5$$

Therefore, (13, 22.62°) in polar coordinates is equivalent to (12, 5) in rectangular coordinates.

$$(13, 22.62°) \Leftrightarrow (12, 5)$$

In general, we can now make the following assumptions:

$$x = r\cos\theta \text{ AND } y = r\sin\theta$$

Example 4

Given: $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $x = r\cos\theta$, AND $y = r\sin\theta$

a. Convert $(7\sqrt{2}, 315°)$

b. Convert (3, 5)

Solution

a. $r = 7\sqrt{2}$ and $\theta = 315°$

$x = r\cos\theta$ $y = r\sin\theta$

$x = (7\sqrt{2})\cos(315°)$ $y = (7\sqrt{2})\sin(315°)$

$x = 7$ $y = -7$

$$(x, y) = (7, -7)$$

b. $x = 3$ and $y = 5$

$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$

$r = \sqrt{3^2 + 5^2}$ $\theta = \tan^{-1}\left(\frac{5}{3}\right)$

$r = \sqrt{34}$ $\theta = 59°$

$r = 5.8$

$$(r, \theta) = (5.8, 59°)$$

Solutions to Scenario Questions

Following are the solutions to the questions posed under "Photonics Scenario Involving Special Graphs."

a. Around 800 nm, loss is just over 2 dB/km

Around 1550 nm, loss is just over 0.2 dB/km

The 1550-nm reading is 10 times smaller than the 800-nm reading.

b. Loss is largest around 1380 nm, a value of just below 20 dB/km, about 100 times the loss at 1550 nm.

Practice Exercises

Exercise 1

In the first example of this section the exponential equation $\frac{V}{V_o} = e^{-t/\tau}$ was graphed in Cartesian coordinates. Graph the same data in semilog, with the *x*-axis (time) on a linear scale and the *y*-axis (voltage ratio) on a logarithmic scale.

Exercise 2

A sound intensity level in decibels (dB) can also be expressed as a logarithmic function of an intensity ratio. The relation is shown below:

$$IL = 10 \log \left(\frac{I}{I_o} \right)$$

where IL = Intensity level (dB)

I/I_o = Sound intensity ratio of actual sound intensity to a reference sound

a. Complete the table below using the formula above:

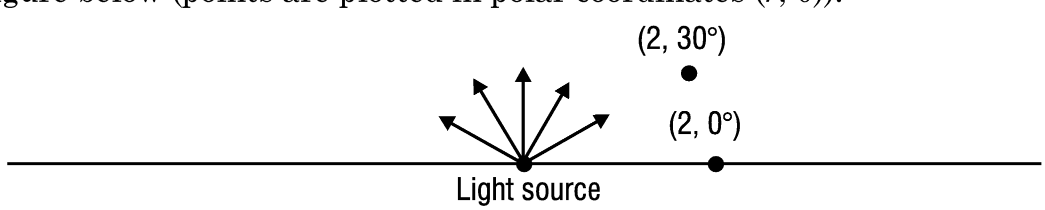
I/I_o Ratio	1	10	20	30	40	50	60	70	80	90	100
Intensity Level (dB)											

b. Graph the above data on a regular scale (two linear axes) with the voltage ratio on the *x*-axis and the intensity level on the *y*-axis.

c. Graph again on a semilog scale (with the *x*-axis on a logarithmic scale and the *y*-axis on a linear scale).

Exercise 3

The area around a point source of light is measured for light intensity as seen in the figure below (points are plotted in polar coordinates (r, θ)).



An instrument that measures light intensity is placed 2 cm, 4 cm, 6 cm, 8 cm, and 10 cm away. At each distance it is placed at 30° increments from 0° to 180°

as demonstrated above. The table below shows the results of these measurements.

Fraction of Original Light Intensity (I_o) at Various Values of the Radius (r) and Degree (θ)

r \ θ	0°	30°	60°	90°	120°	150°	180°
2 cm	0.24	0.25	0.25	0.26	0.25	0.24	0.24
4 cm	0.05	0.06	0.06	0.07	0.06	0.05	0.05
6 cm	0.02	0.03	0.03	0.03	0.03	0.03	0.02
8 cm	0.01	0.02	0.02	0.02	0.02	0.02	0.01
10 cm	0.01	0.01	0.01	0.01	0.01	0.01	0.01

a. Graph the points where light intensity was measured at 2 cm, 6 cm, and 10 cm (at all degrees listed) on a polar coordinate system.

b. What do you notice about the intensities as they relate to the radii and degree measurements?

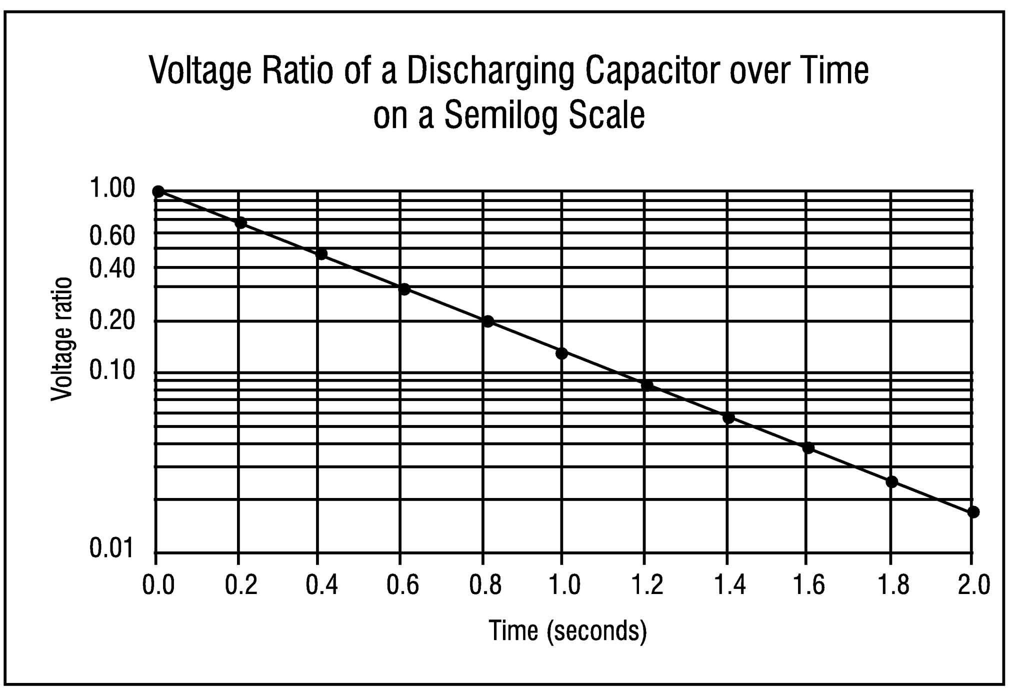
Exercise 4

a. Convert (17, 81°) to rectangular coordinates.

b. Convert (32, 9) to polar coordinates.

Solutions to Practice Exercises

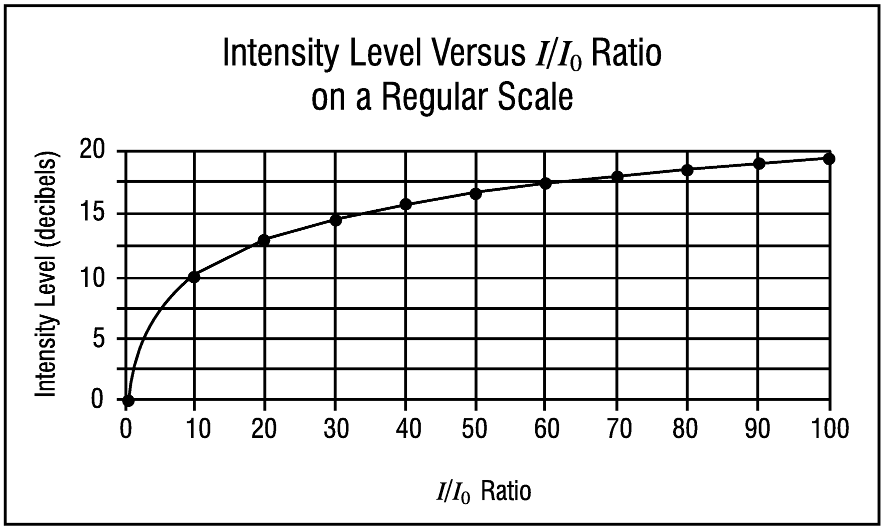
1.



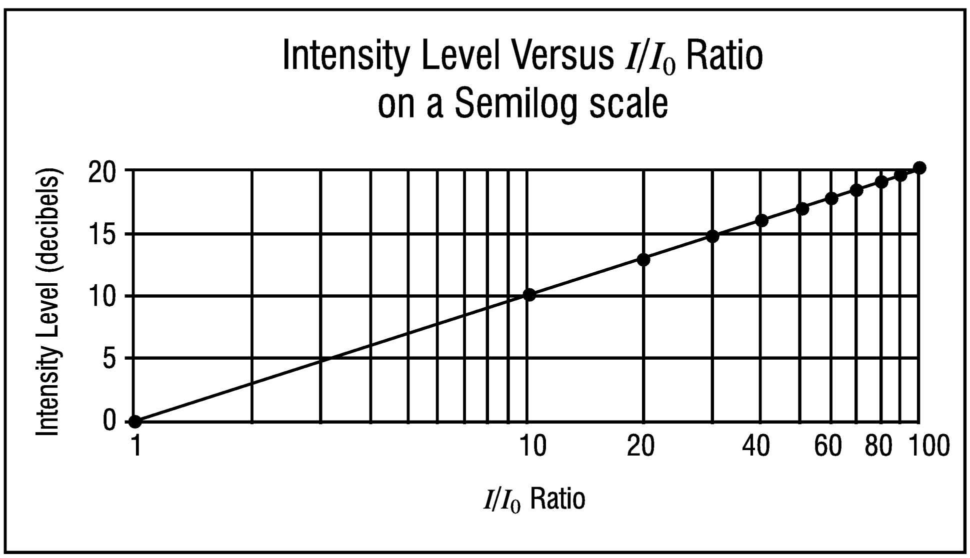
2. a.

I/I_0 Ratio	1	10	20	30	40	50	60	70	80	90	100
Intensity Level (dB)	0	10	13.0	14.8	16.0	17.0	17.8	18.5	19.0	19.5	20.0

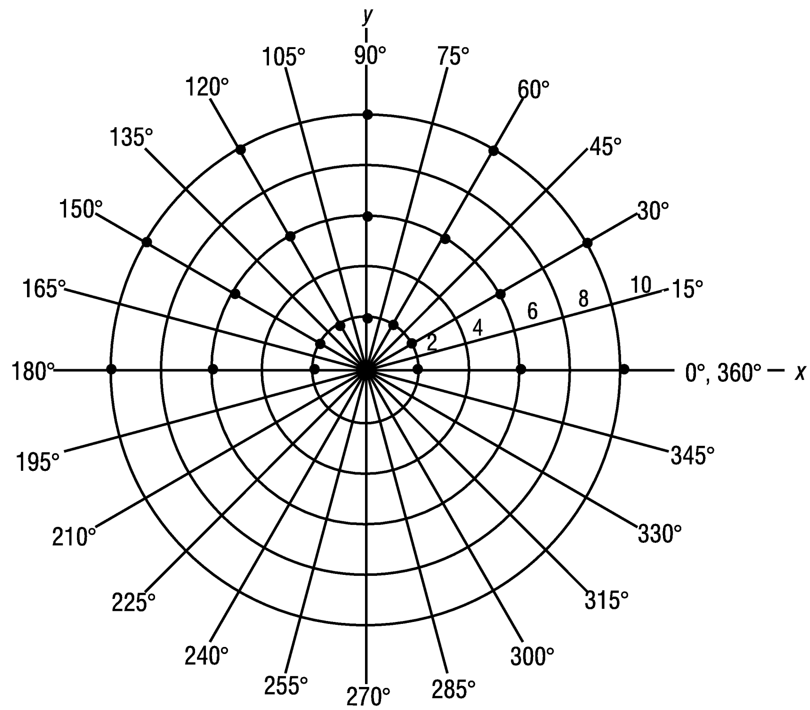
b.



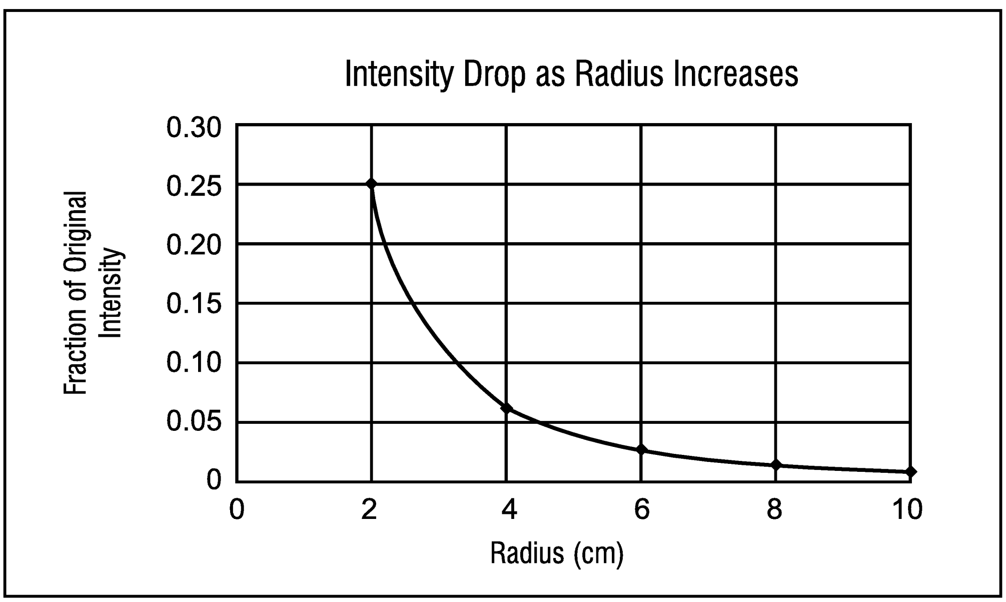
c.



3. a.



b. The intensity decreases as the radius increases (they are inversely proportional) and it is essentially independent of changes in the angle (θ). A more significant graph would use (r, I), where I is the fraction of the original intensity (I_0) remaining.



In this graph, it is clear that as the radius increases the light intensity decreases. In fact, we can now determine a relation between the radius (r) and the percent of original intensity (I/I_0) as follows:

$$I = \frac{I_0}{r^2}$$

4. Given: $r = \sqrt{x^2 + y^2}$, $\theta = \tan^{-1}\left(\frac{y}{x}\right)$, $x = r\cos\theta$ AND $y = r\sin\theta$

a. $(r, \theta) = (17, 81°)$

$r = 17$

$\theta = 81°$

$x = (17)\cos(81°) \qquad y = (17)\sin(81°)$

$x = 2.66 \qquad y = 16.79$

$$(x, y) = (2.66, 16.79)$$

b. $(x, y) = (32, 9)$

$x = 32$

$y = 9$

$r = \sqrt{32^2 + 9^2} \qquad \theta = \tan^{-1}\left(\frac{9}{32}\right)$

$r = 33.24 \qquad \theta = 15.71°$

$$(r, \theta) = (33.24, 15.71°)$$

APPENDIX

Quick Reference of Greek Letters, Constants and Unit Descriptions

Greek Letters

α	Α	Alpha
β	Β	Beta
γ	Γ	Gamma
δ	Δ	Delta
ε	Ε	Epsilon
ζ	Ζ	Zeta
η	Η	Eta
θ	Θ	Theta

ι	Ι	Iota
κ	Κ	Kappa
λ	Λ	Lambda
μ	Μ	Mu
ν	Ν	Nu
ξ	Ξ	Xi
ο	Ο	Omicron
π	Π	Pi

ρ	Ρ	Rho
σ	Σ	Sigma
τ	Τ	Tau
υ	Υ	Upsilon
ϕ, φ	Φ	Phi
χ	Χ	Chi
ψ	Ψ	Psi
ω	Ω	Omega

Physical Constants

Name	Symbol	Approximate Value
Speed of light in vacuum	c	2.997×10^{8} m/s
Permeability of vacuum	μ_o	$4\pi \times 10^{-7}$ N/A^2
Permittivity of vacuum	ε_o	8.85×10^{-12} C^2/N • m^2
Gravitational constant	g	9.81 m/s^2
Avogadro's constant	N_A	6.02×10^{23} mol^{-1}
Boltzmann's constant	k	1.38×10^{-23} J/K
Stefan-Boltzmann's constant	σ	5.67×10^{-8} W/m^2 • K^4
Planck's constant	h	6.63×10^{-34} J • s
Elementary charge	e	1.60×10^{-19} C
Electron rest mass	m_e	9.11×10^{-31} kg
Proton rest mass	m_p	1.67×10^{-27} kg
Neutron rest mass	m_n	1.67×10^{-27} kg
Atomic mass unit	u	1.66×10^{-27} kg

Units

Base Units		
Quantity	**Name**	**Symbol**
Length	Meter	m
Time	Second	s
Mass	Kilogram	kg
Current	Ampere	A
Temperature	Kelvin	K
Amount of substance	Mole	mol

Supplementary Units		
Quantity	**Name**	**Symbol**
Plane Angle	Radian	rad
Solid Angle	Steradian	sr

Derived Units

Unit	Symbol	Unit Details
Newton	N	kg • m/s^2
Joule	J	N • m = kg • m^2/s^2
Watt	W	J/s = kg • m^2/s^3
Hertz	Hz	1/s
Coulomb	C	A • s
Volt	V	J/C = W/A = kg • m^2/(A • s^3)
Ohm	Ω	V/A = kg • m^2/(A^2 • s^3)
Tesla	T	N/(A • m) = kg/(A • s^2)
Weber	Wb	T • m^2 = V • s = J/A = kg • m^2/(A • s^2)
Henry	H	V • s/A = Ω • s = kg • m^2/(A^2 • s^2)
Farad	F	C/V = s/Ω = A^2 • s^4/(kg • m^2)

SI Prefixes

Multiplication Factor	Prefix	Symbol
10^{12}	tera	T
10^{9}	giga	G
10^{6}	mega	M
10^{3}	kilo	k
10^{2}	hecto	h*
10^{1}	deka	da*
10^{-1}	deci	d*
10^{-2}	centi	c
10^{-3}	milli	m
10^{-6}	micro	μ
10^{-9}	nano	n
10^{-12}	pico	p
10^{-15}	femto	f
10^{-18}	atto	a
*Avoid these prefixes when possible.		

Common Identities

Powers and Roots		Exponents and Logarithms
• $b^{\frac{x}{y}} = \sqrt[y]{b^x}$		• $\log_b x = y \Leftrightarrow b^y = x$
• $b^x \cdot b^y = b^{x+y}$	$\leftrightarrow$	• $\log_b(xy) = \log_b x + \log_b y$
• $\frac{b^x}{b^y} = b^{x-y}$	$\leftrightarrow$	• $\log_b\left(\frac{x}{y}\right) = \log_b x - \log_b y$
• $(b^x)^y = b^{xy}$		• $\log_b x^y = y\log_b x$
• $\left(\frac{a}{b}\right)^x = \frac{a^x}{b^x}$		• $\log_{10} x = \log x$
• $b^{-x} = \frac{1}{b^x}$		• $\log_e x = \ln x$
• $\frac{1}{b^{-x}} = b^x$		• $\log_b x = \frac{\ln x}{\ln b} = \frac{\log x}{\log b}$
• $b^x = b^y \Leftrightarrow x = y$	$\leftrightarrow$	• $\log_b x = \log_b y \Leftrightarrow x = y$
• $b^0 = 1$	$\leftrightarrow$	• $\log_b 1 = 0$
• $b^1 = b$	$\leftrightarrow$	• $\log_b b = 1$
		• $\log_b b^x = x = b^{\log_b x}$

Trigonometry		
• $\tan\theta = \frac{\sin\theta}{\cos\theta}$	• $\sin^2\theta + \cos^2\theta = 1$	• $\sin 2\theta = 2\sin\theta\cos\theta$
• $\sin\theta = \frac{1}{\csc\theta}$	• $\tan^2\theta + 1 = \sec^2\theta$	• $\cos 2\theta = \cos^2\theta - \sin^2\theta$
• $\cos\theta = \frac{1}{\sec\theta}$	• $\cot^2\theta + 1 = \csc^2\theta$	• $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
• $\tan\theta = \frac{1}{\cot\theta}$	• $c^2 = a^2 + b^2 - 2ab \cdot \cos C$ • $b^2 = a^2 + c^2 - 2ac \cdot \cos B$ • $a^2 = b^2 + c^2 - 2bc \cdot \cos A$	

Tips for Scientific Calculators

Button	Example	Button Sequence	Answer
sin, cos, tan	cos 60°	[6] [0] [cos]	0.5
$\sin^{-1}$, $\cos^{-1}$, $\tan^{-1}$	$\theta = \sin^{-1}(1)$	[1] [2nd OR INV] [sin]	90°
x^2	27^2	[2] [7] [x^2]	729
$\sqrt{x}$ or $\sqrt{\ }$	$\sqrt{3}$	[3] [2nd OR INV] [$\sqrt{x}$]	1.73
y^x	$53^{1.7}$	[5] [3] [y^x] [1] [.] [7] [=]	853.6
$\sqrt[x]{y}$	$\sqrt[3]{5}$	[5] [2nd OR INV] [$\sqrt[x]{y}$] [3] [=]	1.71
EE or Exp or E	8.6×10^4	[8] [.] [6] [EE] [4]	—
log	log 17	[1] [7] [log]	1.23
ln x or ln	ln 2.4	[2] [.] [4] [ln x]	0.88
(,)	$5 \times (3 + 2)$	[5] [×] [(] [3] [+] [2] [)] [=]	25
π	2π	[2] [×] [2nd OR INV] [π] [=]	6.28
+/– or (–)	$9 \times (-6)$	[9] [×] [6] [+/–] [=]	–54
	3.8×10^{-3}	[3] [.] [8] [EE] [3] [+/–]	—

Conversion Tables

The following tables list the common conversions among various units of measure found in *Principles of Technology*. They are reproduced here for ready reference.

LENGTH

Length	*m*	*km*	*inch*	*ft*	*mi*
1 meter	1	1.0×10^{-3}	39.37	3.281	6.214×10^{-4}
1 kilometer	1000	1	3.937×10^{4}	3281	0.6214
1 inch	0.0254	2.54×10^{-5}	1	0.0833	1.578×10^{-5}
1 foot	0.3048	3.048×10^{-4}	12	1	1.894×10^{-4}
1 mile	1609	1.609	6.336×10^{4}	5280	1

1 angstrom = 10^{-10} m

1 micron (μ) = 10^{-6} m

1 rod = 16.5 ft

1 league = 3 nautical miles

1 nautical mile = 1852 m = 1.1508 mi = 6076.10 ft

1 fathom = 6 ft

1 yard (yd) = 3 ft

1 mil = 10^{-3} inch

MASS

Mass	*g*	*kg*	*lbm*	*slug*	*ton-mass*
1 gram	1	1.0×10^{-3}	2.205×10^{-3}	6.852×10^{-5}	1.102×10^{-6}
1 kilogram	1.0×10^{3}	1	2.205	6.852×10^{-2}	1.102×10^{-3}
1 pound mass	4.536×10^{2}	0.4536	1	3.108×10^{-2}	5.0×10^{-4}
1 slug	1.459×10^{4}	1.459×10^{1}	3.217×10^{1}	1	1.609×10^{-2}
1 ton-mass	9.072×10^{5}	9.07×10^{2}	2.0×10^{3}	6.216×10^{1}	1

1 long ton = 2240 lb = 20 cwt

1 hundredweight (cwt) = 112 lb

1 metric ton = 1000 kg = 2205 lb

1 stone = 14 lb

1 carat = 0.2 g

TIME

Time	*yr*	*day*	*h*	*min*	*s*
1 year	1	3.652×10^{2}	8.766×10^{3}	5.259×10^{5}	3.156×10^{7}
1 day	2.738×10^{-3}	1	24	1.44×10^{3}	8.64×10^{4}
1 hour	1.141×10^{-4}	4.167×10^{-2}	1	60	3.6×10^{3}
1 minute	1.901×10^{-6}	6.944×10^{-4}	1.667×10^{-2}	1	60
1 second	3.169×10^{-8}	1.157×10^{-5}	2.778×10^{-4}	1.667×10^{-2}	1

1 day = period of rotation of earth = 86,164 s
1 year = period of revolution of earth = 365.242 days

FORCE

Force	*dyne*	*kgf*	*N*	*lb*	*pdl*
1 dyne	1	1.020×10^{-6}	1.0×10^{-5}	2.248×10^{-6}	7.233×10^{-5}
1 kilogram force	9.807×10^{5}	1	9.807	2.205	70.93
1 newton	1.0×10^{5}	0.1020	1	0.2248	7.233
1 pound	4.448×10^{5}	0.4536	4.448	1	32.17
1 poundal	1.383×10^{4}	1.410×10^{-2}	0.1383	3.108×10^{-2}	1

PRESSURE

Pressure	*atm*	*inch of water*	*cm Hg*	*N/m^2*	*$lb/inch^2$ (psi)*
1 atmosphere	1	4.068×10^{2}	7.6×10^{1}	1.013×10^{5}	1.470×10^{1}
1 inch of water[a]	2.458×10^{-3}	1	0.1868	2.491×10^{2}	3.613×10^{-2}
1 cm of mercury[a]	1.316×10^{-2}	5.353	1	1.333×10^{3}	0.1934
1 newton per square meter	9.869×10^{-6}	4.105×10^{-3}	7.501×10^{-4}	1	1.450×10^{-4}
1 pound per square inch	6.805×10^{-2}	2.768×10^{1}	5.171	6.895×10^{3}	1

[a]Under standard gravitational acceleration and temperature of 4°C for water, 0°C for mercury

1 bar = 1.013×10^{5} N/m^2
1 cm of water = 98.07 N/m^2
1 torr = 1 mm of Hg
1 ft of water = 62.43 lb/ft^2

1 atm = 29.92 in. of Hg
1 atm = 33.92 ft of water
1 atm = 2117 lb/ft^2

ENERGY

Energy	*Btu*	*ft•lb*	*J*	*kcal*	*kWh*
1 British thermal unit	1	7.779×10^{2}	1.055×10^{3}	0.2520	2.930×10^{-4}
1 foot-pound	1.285×10^{-3}	1	1.356	3.240×10^{-4}	3.766×10^{-7}
1 joule	9.481×10^{-4}	0.7376	1	2.390×10^{-4}	2.778×10^{-7}
1 kilocalorie	3.968	3.086×10^{3}	4.184×10^{3}	1	1.163×10^{-3}
1 kilowatt-hour	3.413×10^{3}	2.655×10^{6}	3.6×10^{6}	8.602×10^{2}	1

1 Btu = 252 cal
1 Btu = 778 ft • lb
1 Btu = 1055 J
1 J = 0.239 cal
1 cal = 3.09 ft • lb
1 cal = 4.18 J
1 kcal = 1000 cal
1 ft • lb = 0.324 cal

POWER

Power	*Btu/h*	*ft•lb/s*	*hp*	*kcal/s*	*W*
1 Btu/h	1	0.2161	3.929×10^{-4}	7.0×10^{-5}	0.2930
1 ft•lb/s	4.628	1	1.818×10^{-3}	3.239×10^{-4}	1.356
1 horsepower	2.545×10^{3}	5.50×10^{2}	1	0.1782	7.457×10^{2}
1 kcal/s	1.429×10^{4}	3.087×10^{3}	5.613	1	4.184×10^{3}
1 watt	3.413	0.7376	1.341×10^{-3}	2.390×10^{-4}	1

1 ton refrigeration = 12,000 Btu/h

SPEED

Speed	*ft/s*	*km/h*	*m/s*	*mi/h*	*knot*
1 foot per s	1	1.097	0.348	0.6818	0.5925
1 kilometer per h	0.9113	1	0.2778	0.6214	0.5400
1 meter per s	3.281	3.6	1	2.237	1.944
1 mile per h	1.467	1.609	0.4470	1	0.8689
1 knot	1.688	1.852	0.5144	1.151	1

1 knot = 1 nautical mile/h

MASS-ENERGY EQUIVALENTS

Mass-Energy Equivalents	*kg*	*amu*	*J*	*MeV*
1 kilogram	1	6.02×10^{26}	8.987×10^{16}	5.610×10^{29}
1 atomic mass unit	1.666×10^{-27}	1	1.492×10^{-10}	9.315×10^{2}
1 joule	1.113×10^{-17}	6.68×10^{9}	1	6.242×10^{12}
1 million electron volts	1.783×10^{-30}	4.17×10^{22}	1.602×10^{-13}	1

WAVELENGTH CONVERSIONS

λ	Å (angstrom)	nm (nanometer)	μm (micrometer)	cm (centimeter)	m (meter)
1 Å	1	10^{-1}	10^{-4}	10^{-8}	10^{-10}
1 nm	10	1	10^{-3}	10^{-7}	10^{-9}
1 μm	10^{4}	10^{3}	1	10^{-4}	10^{-6}
1 cm	10^{8}	10^{7}	10^{4}	1	10^{-2}
1 m	10^{10}	10^{9}	10^{6}	10^{2}	1